

HÖJDPUNKTEN 2026

Open Competition, 2X March 2026

Solutions

Problem 1. A grocery store has a sale: Buy at least four fruits and get the cheapest one for free. The first customer buys three oranges and one banana. The second one buys three oranges and two lemons. The third one buys two bananas and two lemons. It turns out that all three customers paid exactly 35 kr for their fruits. Find the price for each individual fruit.

Solution. Let a , b and c denote the unit prices of oranges, bananas and lemons, respectively. If $c \geq b$, customer 2 would have paid more than customer 1, which they did not, so we must have $b > c$. If $a \geq b$, customer 2 would have paid more than customer 3, so we must have $b > a$. We conclude that the bananas are the most expensive of the three fruits. It remains to determine whether a or c is larger. Customers 1 and 3 pay $2a + b$ and $b + 2c$ respectively for their fruit. If $c > a$, then

$$35 = 2a + b < 2c + b < b + 2c = 35,$$

which is impossible. Hence $a \geq c$. We have now determined the price ordering of the fruits, so we know how many fruits of each kind each customer must pay for. This gives the system of equations

$$\begin{cases} 2a + b = 35 & (1) \\ 3a + c = 35 & (2) \\ 2b + c = 35 & (3) \end{cases}$$

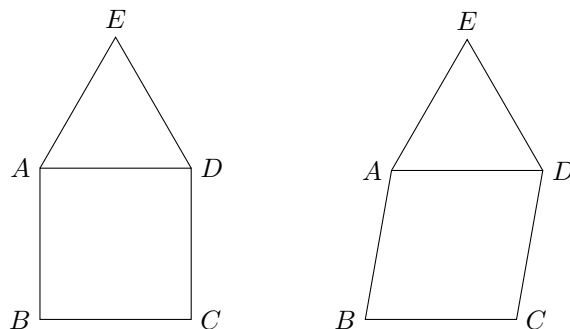
From (1) and (2) we get $b = 35 - 2a$ and $c = 35 - 3a$. Substituting into (3) gives

$$\begin{aligned} 2(35 - 2a) + (35 - 3a) &= 35 \\ \iff 105 - 7a &= 35 \\ \iff a &= \frac{105 - 35}{7} = 10. \end{aligned}$$

Hence $b = 35 - 20 = 15$ and $c = 35 - 30 = 5$.

Answer: The oranges cost 10 kr, the bananas cost 15 kr and the lemons cost 5 kr. \square

Problem 2. The picture shows a two dimensional model of a house consisting of six line segments of equal length. A strong gust of wind has made it so that the walls of the house (AB and CD) have started to tilt (by an angle smaller than 30°). Prove that the angle $\angle BEC$ does not depend on the tilt and calculate the size of this angle.



Solution. Let $x = \angle AEB$, $y = \angle BEC$ and $z = \angle CED$. The triangle $\triangle ADE$ is equilateral, so

$$x + y + z = \angle AED = 60^\circ. \quad (*)$$

We also have

$$x = \angle ABE = \angle ABC - \angle EBC,$$

$$z = \angle DCE = \angle DCB - \angle ECB.$$

From the angle sum in triangle $\triangle BCE$ we get

$$y = 180^\circ - \angle EBC - \angle ECB,$$

hence

$$x + z - y = \angle ABC + \angle DCB - 180^\circ = 0 \quad (**)$$

where we used the fact that the sum of two adjacent angles in a rhombus ($\square ABCD$) is 180° . Finally, subtracting $(**)$ from $(*)$ gives

$$2y = 60^\circ,$$

i.e. $y = 30^\circ$. □

Problem 3. Find all solutions to the system of equations

$$\begin{cases} p + q + r = s \\ p + 2q + 3r = 5t \end{cases}$$

where p, q, r, s and t are prime numbers.

Solution. Subtracting the first equation from the second, we get

$$q + 2r = 5t - s.$$

Both s and t are obviously greater than 2, hence odd, so the right-hand side is odd $-$ odd $=$ even. This means q is even, i.e. $q = 2$.

From the first equation we now see that $p + r$ must be odd. This gives two cases:

Case 1 $p = 2$ and r odd: The second equation becomes

$$2 + 4 + 3r = 5t.$$

But the left-hand side is a multiple of 3, so $t = 3$, and hence $r = (15 - 4 - 2)/3 = 3$. The first equation then gives $s = 7$. Substituting back, we see that this solves the system.

Case 2 p odd and $r = 2$: The second equation now becomes

$$p + 4 + 6 = 5t \iff p = 5t - 10.$$

The right-hand side is now a multiple of 5, so $p = 5$ and $t = 3$. But substituting this into the first equation gives $s = 5 + 2 + 2 = 9$, which is not prime.

Answer: The system has only one solution, namely $(p, q, r, s, t) = (2, 2, 3, 7, 3)$. \square

Problem 4. Let n be a positive integer. In the old castle, the royal glazier is working to install a new window in the throne room. She must place $\frac{1}{2}n(n+1)$ purple glass panes and $\frac{1}{2}n(n-1)$ yellow glass panes in a $n \times n$ grid of panes such that no two rows have the same number of purple panes and no two columns have the same number of purple panes. In how many ways can she do this?

Solution. Note that no row/column can have 0 purple panes, since the remaining $n-1$ rows/columns would then contain at most $\sum_{i=2}^n i = \frac{1}{2}n(n+1) - 1 < \frac{1}{2}n(n+1)$ purple panes. Hence the rows must contain $1, \dots, n$ purple panes in some order, and similarly for the columns.

Note that any window satisfying the conditions can have its rows and columns permutable arbitrarily, and the resulting window would still satisfy the conditions. We claim that the number of windows satisfying the conditions equals $(n!)^2$. Therefore it suffices to show that there is exactly one window which has i purple panes in the i -th column from the left and j purple panes in the j -th row from the bottom, for every $1 \leq i, j \leq n$.

By induction on k it can be shown that column $n+1-k$ has its bottom $n+1-k$ panes purple and its top $k-1$ panes yellow. The base case $k=1$ is trivial, since column n must have n purple panes. The induction step follows from the fact that the top $k-1$ rows already have enough purple panes in columns $\{n+2-k, \dots, n\}$, and so the top $k-1$ panes in column $n+1-k$ must be yellow, which forces the bottom $n+1-k$ panes to be purple.

Therefore there are exactly $(n!)^2$ ways for the royal glazier to install the new window. \square

Problem 5. Let $\triangle ABC$ be a right-angled triangle with $\angle ABC = 90^\circ$ and $|AB| < |BC|$. Let D be a point on the hypotenuse AC such that $|AB| = |BD|$. The point T lies on side BC and is such that $\angle ATB = \angle CTD$. Prove that the line through D perpendicular to BD splits the segment CT in half.

Solution. Let A' be the reflection of the point A across the line BC . Then $\angle A'TB = \angle ATB = \angle CTD$, which implies that the points A' , T and D are collinear.

Since $|BA| = |BD| = |BA'|$, we have $\angle A'DA = 90^\circ$ by Thales' theorem. Since $\angle ABT + \angle TDA = 90^\circ + 90^\circ = 180^\circ$, the quadrilateral $ABTD$ is cyclic.

Let M be the midpoint of the segment CT . Since $\angle CDT = 90^\circ$, Thales' theorem gives $|MC| = |MD| = |MT|$. Finally, note that

$$\angle MDB = \angle MDT + \angle TDB = \angle DTC + \angle TAB = \angle BTA + \angle TAB = 90^\circ,$$

so the line through D perpendicular to BD bisects the segment CT , as desired. \square

Problem 6. A colouring of the cells of a $n \times n$ grid in the colours red and blue is called *elegant* if it is possible to move between each pair of red cells, by moving between cells with a common side, without having to visit any third red cell. Let $R(n)$ be the largest possible number of red cells that an *elegant* colouring of a $n \times n$ grid can have. Determine

$$\lim_{n \rightarrow \infty} \frac{R(n)}{n^2}.$$

Solution. We claim that the answer is $\frac{2}{3}$.

Lemma 6.1. For every n , $R(n) \geq (n-1) \left\lceil \frac{2(n-2)}{3} \right\rceil$.

Proof. Let us label the cells of the grid by coordinates (x, y) , where x denotes the column number and y the row number, $0 \leq x \leq n-1$, $0 \leq y \leq n-1$. We claim that we obtain an elegant colouring by colouring the cell (x, y) as follows:

$$\begin{cases} \text{blue} & \text{if } y = 0, x = 0, x = n-1, \text{ or } x \text{ is divisible by } 3, \\ \text{red} & \text{otherwise.} \end{cases}$$

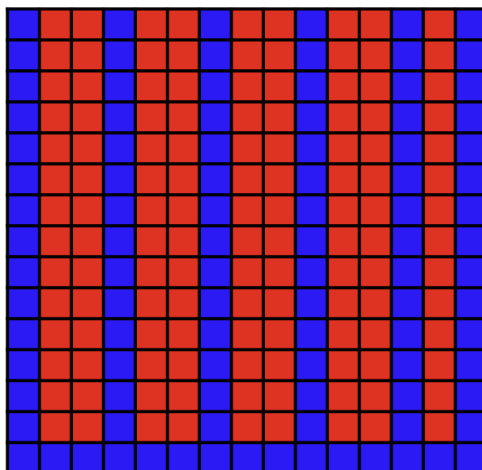


Figure 1: An elegant colouring with $(n-1) \left\lceil \frac{2(n-2)}{3} \right\rceil$ red cells for $n = 15$.

This colouring is elegant because from any red cell one can step either right or left one step, and then down to the bottom row, which consists only of blue cells. From the bottom row one can in the same way reach any other red cell, having only passed through blue cells. We now claim that this colouring uses at least $\frac{2}{3}(n-1)(n-2)$ red cells. Indeed, there are $\left\lceil \frac{2(n-2)}{3} \right\rceil$ columns with red cells in this colouring, and each such column contains $n-1$ red cells, giving $(n-1) \left\lceil \frac{2(n-2)}{3} \right\rceil$ red cells in total. There may be elegant colourings with more red cells, but the colouring we have found shows that $R(n)$ must be at least $(n-1) \left\lceil \frac{2(n-2)}{3} \right\rceil$. \square

Lemma 6.2. For every n , $R(n) \leq \frac{2}{3}n^2 + 2$.

Proof. Suppose we have an elegant colouring of a $n \times n$ grid. The lemma is easily seen to hold for $n = 1$ or $n = 2$. From now on we assume $n \geq 3$, and we split into two cases.

- (i) All red cells have at least two blue neighbours. Let E_{RB} be the number of edges between red cells and blue cells. From our assumption we have $E_{RB} \geq 2r$, where r is the number of red cells. At the same time $E_{RB} \leq 4b$, where b is the number of blue cells, since every blue cell has at most 4 red neighbours. Hence $4b \geq 2r \iff 2b \geq r$. This means at most two thirds of all cells can be red, i.e. $r \leq \frac{2}{3}n^2$.
- (ii) There is a red cell with exactly one blue neighbour. Say the red cell has coordinates (x_0, y_0) . Since the colouring is elegant, one must be able to move from (x_0, y_0) to all red cells without passing through any third red cell. One can of course go directly from (x_0, y_0) to its 3 red neighbours without passing through any third red cell, but to reach any other red cell one must pass through the only blue neighbour of (x_0, y_0) . The blue neighbour must therefore be part of a connected region (one can move between any two cells in the region without leaving the region) of blue cells adjacent to all red cells except possibly the 3 red neighbours of (x_0, y_0) . Let us call this region B and suppose it contains b_1 blue cells. Let E be the number of edges between blue cells in B . Since B is connected, $E \geq b_1 - 1$, a well-known result in graph theory. We now claim $2E + r - 3 \leq 4b_1$. For a cell z , let $g_b(z)$, $g_r(z)$, $g(z)$ denote the number of blue, red and total neighbours of z , respectively. Then

$$2E + r - 3 \leq \sum_{z \in B} g_b(z) + \sum_{z \in B} g_r(z) = \sum_{z \in B} g(z) \leq 4b_1.$$

Thus

$$2(b_1 - 1) + r - 3 \leq 4b_1 \implies r \leq 2b_1 + 5 \leq 2b + 5,$$

where b is the total number of blue cells. Adding $2r$ to both sides and using $r + b = n^2$, we get

$$3r \leq 2n^2 + 5 \iff r \leq \frac{2}{3}n^2 + \frac{5}{3} \leq \frac{2}{3}n^2 + 2.$$

- (iii) Some red cell has no blue neighbour. Then there can be at most 5 red cells, since the red cell with no blue neighbours must be adjacent to all the other red cells. For $n \geq 3$ we have $5 \leq \frac{2}{3}n^2 + 2$.

We have shown that there can never be more than $\frac{2}{3}n^2 + 2$ red cells in an elegant colouring of an $n \times n$ grid, which proves the lemma. \square

From Lemmas 6.1 and 6.2 it follows immediately that

$$\lim_{n \rightarrow \infty} \frac{R(n)}{n^2} = \frac{2}{3}.$$

\square

Problem 7. In Numberland, each city is named after a positive integer. Two citizens can only exchange letters with each other if they live in different cities whose names differ by a power of two (note that two citizens living in the same city cannot exchange letters). Given that Numberland has n citizens, what is the largest possible number of pairs of citizens in Numberland that can exchange letters with each other?

Solution. Let $f(n)$ denote the largest possible number of pairs of inhabitants of Numberland that can correspond. We claim that $f(n) = \lfloor \frac{n^2}{3} \rfloor$.

Lemma 7.1. $f(n) \geq \lfloor \frac{n^2}{3} \rfloor$.

Proof. Place $\lfloor \frac{n}{3} \rfloor$ inhabitants in city 1, $\lfloor \frac{n+1}{3} \rfloor$ inhabitants in city 2, and $\lfloor \frac{n+2}{3} \rfloor$ inhabitants in city 3. \square

Lemma 7.2. $f(n) \leq \lfloor \frac{n^2}{3} \rfloor$.

Proof. Suppose the inhabitants of Numberland have settled so that the number of pairs that can exchange letters is maximal. Let A and B be two inhabited cities whose inhabitants cannot exchange letters with each other, where an inhabitant of city A can exchange letters with at least as many other inhabitants of Numberland as an inhabitant of city B can. If all inhabitants of city B move to city A , the total number of pairs that can exchange letters does not decrease.

Let us move the inhabitants of Numberland in this way until exchanging letters is possible between every pair of inhabited cities. We claim that after this there can be at most 3 inhabited cities in Numberland.

Suppose for contradiction that 4 or more inhabited cities remain such that exchanging letters is possible between every pair. Let four of these cities have names $a > b > c > d$. Then $a - b$, $a - c$, $a - d$, $b - c$, $b - d$ and $c - d$ are all powers of two. Since the sum of two powers of two is itself a power of two if and only if the two summands are equal, we obtain

$$\begin{aligned} (a - b) + (b - c) = (a - c) &\implies (a - b) = (b - c) \\ (b - c) + (c - d) = (b - d) &\implies (b - c) = (c - d) \\ (a - c) + (c - d) = (a - d) &\implies (a - c) = (c - d). \end{aligned}$$

But this would imply $(c - d) = (a - c) = (a - b) + (b - c) = 2(c - d)$, a contradiction (since $c > d$).

Hence we may move the inhabitants of Numberland, without decreasing the number of pairs that can exchange letters, until they live in at most 3 different cities. Let these three cities have x , y and z inhabitants. Then there are at most

$$xy + yz + zx \leq \frac{1}{3}(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) = \frac{1}{3}(x + y + z)^2 = \frac{n^2}{3}$$

pairs of inhabitants that can exchange letters.

Therefore $f(n) \leq \frac{n^2}{3} \implies f(n) \leq \lfloor \frac{n^2}{3} \rfloor$. \square

We have shown that $\lfloor \frac{n^2}{3} \rfloor \leq f(n) \leq \lfloor \frac{n^2}{3} \rfloor \implies f(n) = \lfloor \frac{n^2}{3} \rfloor$. \square

Problem 8. Matilda has an odd prime p and a grid with n rows, where n is a positive integer. On the k 'th row ($1 \leq k \leq n$) she enters the digits of the number kn in base p , with the i 'th digit (from the right) of the number being entered into column i for all i .

It turns out that if Matilda chooses any column of the grid and sums all digits entered into that column, the result will always be divisible by p . Prove that p divides n .

Solution. Note that the sum of the digits in column $\ell + 1$ is congruent modulo p to the following expression:

$$\sum_{k=1}^n \left\lfloor \frac{kn}{p^\ell} \right\rfloor.$$

Suppose that $p \nmid n$. We will show that $p^m \mid n + 1$ by induction on m . The base case $m = 0$ is trivial. For the induction step, suppose $p^m \mid n + 1$. Then

$$\begin{aligned} \sum_{k=1}^n \left\lfloor \frac{kn}{p^m} \right\rfloor &= \sum_{k=1}^n \left\lfloor \frac{k(n+1)}{p^m} - \frac{k}{p^m} \right\rfloor \\ &= \sum_{k=1}^n \frac{k(n+1)}{p^m} - \left\lfloor \frac{k}{p^m} \right\rfloor \\ &= \frac{n(n+1)^2}{2p^m} + \frac{n+1}{p^m} - \sum_{k=1}^{n+1} \left\lfloor \frac{k}{p^m} \right\rfloor \\ &= \frac{n(n+1)^2}{2p^m} + \frac{n+1}{p^m} - p^m \left[\frac{\left(\frac{n+1}{p^m}\right) \left(\frac{n+1}{p^m} + 1\right)}{2} \right] \\ &= \frac{1}{2} \left(\frac{n+1}{p^m} \right) (n^2 + 1 - p^m) \end{aligned}$$

must be divisible by p .

If $\frac{n+1}{p^m}$ is divisible by p , the induction step is complete. Otherwise, $n^2 + 1 - p^m$ must be divisible by p . If $m = 0$ this gives $p \mid n^2$, a contradiction. If $m \geq 1$, then $n^2 + 1 - p^m$ is congruent to 2 modulo p , and hence cannot be divisible by p . This proves the induction step.

Hence $p^m \mid n + 1$ for every non-negative integer m , which contradicts the assumption $p \nmid n$. Therefore n must be divisible by p , which is what we wanted to show. \square

Problem 9. Ulrich wishes to synchronize his n clocks which have all stopped, where n is an odd positive integer. All of Ulrich's clocks have a 12-hour clock face with an hour hand and a minute hand. Ulrich can adjust each clock's minute hand directly at a speed of 10 revolutions per minute. In the worst case, how long must Ulrich spend adjusting minute hands if he chooses the time to which he synchronizes his clocks to optimally?

Solution. We claim that the answer is $0.3(n - \frac{1}{n})$ min.

Let the clock times be $a_1, \dots, a_n \in S^1$ in that order. If Ulrich synchronizes the clocks to the time T , this will take

$$1.2 \text{ min} \cdot \sum_{i=1}^n |T - a_i|,$$

where $|x - y|$ is the distance between x and y on S^1 .

Note that $\mathbb{E}_k[|a_{k+i} - a_k|] \leq \frac{|i|}{n}$ for every integer i . Hence

$$\mathbb{E}_k \left[1.2 \text{ min} \cdot \sum_{i=1}^{\frac{n-1}{2}} (|a_{k+i} - a_k| + |a_{k-i} - a_k|) \right] \leq 1.2 \text{ min} \cdot \sum_{i=1}^{\frac{n-1}{2}} \frac{2i}{n} = 0.3 \left(n - \frac{1}{n} \right) \text{ min},$$

so for some clock time a_k it takes Ulrich at most $0.3(n - \frac{1}{n})$ min to synchronize all his clocks to that time.

To show that Ulrich cannot do better, consider the case when a_1, \dots, a_n are spread out evenly around S^1 . Suppose Ulrich synchronizes to the time T . Consider the two times a_k and a_{k+1} that are closest to $-T$. Then $|T - a_{k+1-i}| + |T - a_{k+i}| = \frac{n+1-2i}{n}$ for every $1 \leq i \leq \frac{n-1}{2}$, so the time it takes to synchronize to T is

$$1.2 \text{ min} \cdot \sum_{i=1}^n |T - a_i| \geq 1.2 \text{ min} \cdot \sum_{i=1}^{\frac{n-1}{2}} \frac{n+1-2i}{n} = 0.3 \left(n - \frac{1}{n} \right) \text{ min}.$$

□

Problem 10. Let $\triangle ABC$ be an acute triangle with orthocenter H . Let Γ be the circle passing through H which is tangent to the circumcircle of triangle $\triangle ABC$ at A . Let M be the center of Γ . Suppose that Γ intersects lines BH and CH again at points D and E , respectively.

Prove that the circumcircle of triangle $\triangle MDE$ is tangent to line BC .

Solution. This solution uses directed angles.

Let O denote the centre of the circumscribed circle of triangle ABC . Let $N = AO \cap BC$ and $K = AO \cap \Gamma \neq A$. Then AK is a diameter of Γ , so $AH \perp HK$, which implies $HK \parallel BC$. Hence

$$\angle ANB = \angle AKH = \angle ADH = \angle ADB,$$

so A, D, N and B are concyclic. By symmetry, A, E, N and C are also concyclic.

Furthermore, since AH and AO are isogonal in $\angle BAC$, we have

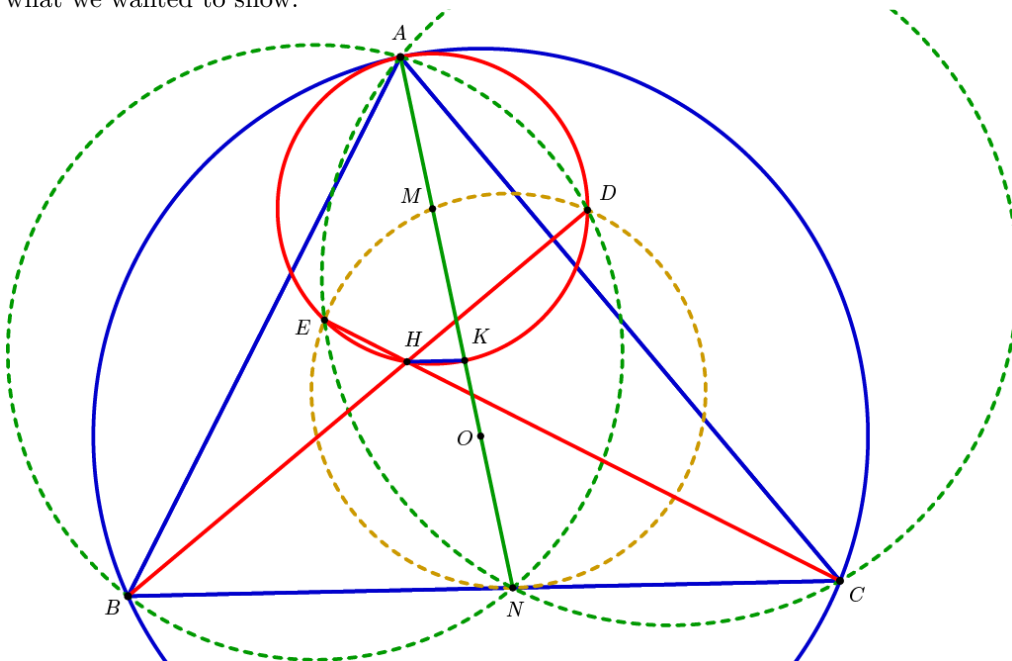
$$\begin{aligned} \angle NBD &= \angle KHD = \frac{\pi}{2} - \angle DHA \\ &= \frac{\pi}{2} - \angle BHA \\ &= \angle HAC = \angle BAO = \angle BAN = \angle BDN, \end{aligned}$$

so $|BN| = |DN|$, and by symmetry $|CN| = |EN|$.

Now

$$\angle ENB = 2\angle ECN = 2\angle EAN = 2\angle EAM = \angle EMN,$$

which means the circumscribed circle of triangle MEN is tangent to the line BC at N , where the last step uses that $|AM| = |EM|$. By symmetry, the circumscribed circle of triangle MDN is also tangent to the line BC at N . From this it follows that D, M, E and N are concyclic and that their circumscribed circle is tangent to BC at N , which is what we wanted to show.



□

Problem 11. Scott has an infinite sheet of paper with a grid of equilateral triangles. He cuts out a connected figure by only cutting along the gridlines. Then, he folds the figure in the following way: each fold occurs along a gridline and results in the area of the shape being halved. After a finite number of folds, he is left with a triangle cell. Prove that what was left of the sheet of paper after he cut out his figure is connected.

[By a connected part of the grid we mean a set of triangle cells such that it is possible to move between each pair of cells in this set by only moving between cells in the set with common sides.]

Solution. Let n be the total number of folds Scott performs. Let F_n be the paper figure Scott began with, and let F_{i-1} be the paper figure obtained when he folds F_i over the line ℓ_i for each $1 \leq i \leq n$. The fact that the area of F_{i-1} is half the area of F_i implies that F_i must consist of two copies of F_{i-1} that are mirror images of each other across the line ℓ_i .

Suppose n is minimal such that what remained of the infinite sheet of paper after Scott cut out the figure F_n is not connected. Then F_n must have (at least) one hole. Since F_{n-1} has no holes, ℓ_n must pass through this hole; hence there must exist two triangle cells A and B in F_{n-1} such that:

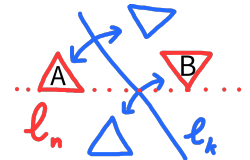
- (i) A has a side on ℓ_n ;
- (ii) B has a vertex (or a side) on ℓ_n ;
- (iii) A and B do not touch each other; and
- (iv) the triangle cell next to B that touches ℓ_n and lies closer to A (than B does) is not in F_{n-1} ;

since F_n is connected, and ℓ_n passes through a hole in F_n .

Since Scott eventually folds F_{n-1} down to a single triangle cell, there will be a fold line ℓ_k that has A and B on opposite sides. Then we have two cases:

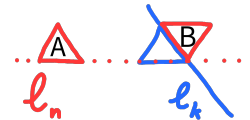
- B does not have a side on the line ℓ_k .

Then the reflection of either A or B across the line ℓ_k lies on the other side of the line ℓ_n . This is a contradiction since F_k must lie entirely on one side of ℓ_n while having reflective symmetry across ℓ_k .



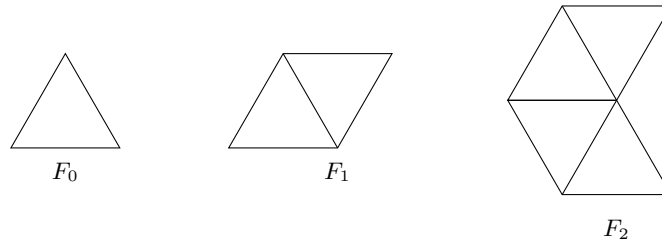
- B has a side on the line ℓ_k .

Then the reflection of B across the line ℓ_k lies closer to A than B does, contradicting (iv).

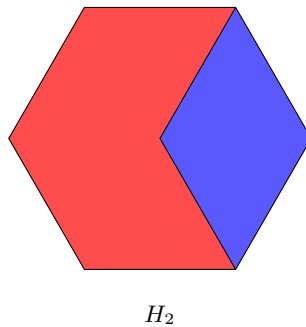


Hence the assumption above is false, so we have shown that what remains after Scott cuts out his paper figure must be connected. \square

Alternative Solution. We use the same notation as in the solution above. Consider what happens if Scott unfolds the paper again. He then begins with a single equilateral triangle F_0 . Furthermore, F_1 must be a rhombus consisting of two copies of F_0 , and F_2 must be the figure shown below.



Now consider, for $k \geq 2$, the smallest convex polygon H_k whose sides are parallel to the gridlines of the triangular grid, and which contains the figure F_k . For example, H_2 is a regular hexagon of side length 1 (see figure).



We want to show by induction that the following holds for $k \geq 2$:

- For each side of H_k , its intersection with F_k is either a single point or a single connected segment.
- H_{k+1} is a hexagon obtained by reflecting H_k across one of its sides s , and then extending the two sides which are not parallel to s and which do not share a vertex with s , as well as extending their reflections.

The base case consists of showing the first point for $k = 2$, which we have already done (see the picture of H_2 above).

For the induction step, since F_k lies entirely on one side of the line ℓ_{k+1} , the line ℓ_{k+1} must be an extension of a side s of H_k . Reflecting H_k across the side s , every side of H_k and its reflection H'_k intersects F_{k+1} in either a single point or a single connected segment. This remains true when the two sides of H_k (and H'_k) which are not parallel to s and which do not share a vertex with s are extended to form a hexagon Q together with the side of H_k (and H'_k) parallel to s but not coinciding with s . Note that no strictly smaller convex polygon than Q can contain all the points and sides of F_{k+1} that lie on the sides of Q , so $H_{k+1} = Q$. \square

Problem 12. An infinite sequence r_0, r_1, r_2, \dots , consists of rational numbers, and for each integer $n \geq 1$ it holds that r_n is a root of the polynomial

$$x^n + r_{n-1}x^{n-1} + \dots + r_0.$$

Prove that there exists a number N such that $r_n = r_N$ for all $n > N$.

Solution. We first note that $r_0 = 0$ gives $r_n = 0$ for all n , which can be shown by induction. The sequence $0, 0, 0, \dots$ is constant, in which case we may take $N = 0$. Note also that if $r_0 \neq 0$, then no element of the sequence can equal 0, since 0 is a root of $x^n + r_{n-1}x^{n-1} + \dots + r_0$ only if $r_0 = 0$. From now on we assume $r_0 \neq 0$. Define the polynomial

$$f_n(x) = x^n + \sum_{k=0}^{n-1} r_k x^k.$$

The condition from the problem statement is then that $f_n(r_n) = 0$ for all $n \geq 1$. For each n we write $r_n = \frac{a_n}{b_n}$, where a_n and b_n are coprime integers and $a_n > 0$.

Lemma 12.1. For every n , $a_n \mid a_0$ and $b_n \mid b_0$.

Proof. The lemma clearly holds for $n = 0$, and we assume inductively that it holds for $n = 0, 1, \dots, n-1$. We have that r_n is a root of f_n , and hence also of $b_0 f_n$. We have

$$b_0 f_n(x) = b_0 x^n + b_0 r_{n-1} x^{n-1} + \dots + b_0 r_0.$$

By the induction hypothesis, every coefficient of the polynomial above is an integer, since $b_m \mid b_0$ for all $m < n$. The rational root theorem now gives $b_n \mid b_0$ and $a_n \mid a_0$, since $b_0 r_0 = a_0$. The lemma now follows by induction. \square

From the lemma we get that the set $\{r_n \mid n \in \mathbb{N}\}$ is finite, since a_0 and b_0 have only finitely many divisors. Each number in the sequence $(r_n)_{n=0}^\infty$ either appears finitely many times or infinitely often, and there must exist an $N \geq 1$ such that r_n appears infinitely often whenever $n \geq N$.

Every number that appears infinitely often appears at least 2 times. Suppose $r_n = r_{n+k}$, where $n \geq N$ and $k \geq 1$. Define

$$g_{n,k}(x) := x^{1-n}(f_{n+k}(x) - f_n(x)) = x^k - 1 + \sum_{m=0}^{k-1} r_{n+m} x^m.$$

Since $r_n \neq 0$ and $f_n(r_n) = f_{n+k}(r_n) = 0$, we have $g_{n,k}(r_n) = 0$.

Lemma 12.2. For $n \geq N$, $a_n = 1$.

Proof. Note that the polynomial $b_0 \cdot g_{n,k}(x)$ has integer coefficients, and its constant coefficient is $b_0 r_n - b_0 = \frac{b_0}{b_n} a_n - b_0$. Since r_n is a root of $g_{n,k}$, we have $a_n \mid \frac{b_0}{b_n} a_n - b_0 \Rightarrow a_n \mid b_0$. Since $a_n \mid a_0$ and $a_n \mid b_0$, we must have $a_n = 1$. Therefore, for all $n \geq N$, $r_n = \frac{1}{b_n}$. \square

Substituting $a_n = 1$ into the equation $g_{n,k}(r_n) = 0$ (again with $n \geq N$), we get

$$1 - \frac{1}{b_n} = b_n^{-k} + \sum_{m=1}^{k-1} r_{n+m} b_n^{-m} \tag{1}$$

for all $n \geq N$.

Lemma 12.3. For $n \geq N$, $b_n \in \{-1, 1, 2\}$.

Proof. Suppose $b_n \notin \{-1, 1, 2\}$ for some $n \geq N$. Applying the triangle inequality to (1), we get

$$\left|1 - \frac{1}{b_n}\right| = \left|b_n^{-k} + \sum_{m=1}^{k-1} r_{n+m} b_n^{-m}\right| \leq |b_n|^{-k} + \sum_{m=1}^{k-1} |r_{n+m}| |b_n|^{-m}.$$

Note that $|r_{n+m}| = |b_{n+m}^{-1}| \leq 1$ for all $m > 0$. We have $|b_n|^{-1} < 1$, and using the geometric series we obtain

$$\left|1 - \frac{1}{b_n}\right| < \sum_{m=1}^{\infty} |b_n|^{-m} = \frac{1}{|b_n| - 1}.$$

This can in turn be rewritten as

$$|b_n - 1| \cdot (|b_n| - 1) < |b_n|.$$

However, b_n cannot satisfy the inequality above if $b_n \notin \{-1, 1, 2\}$. This can be proved by induction: the inequality fails for $b_n = -2$ and $b_n = 3$, and as we increase or decrease b_n by 1, the left-hand side grows faster than the right-hand side. So the only values b_n can take for $n \geq N$ are $-1, 1, 2$. \square

Lemma 12.4. *If $b_n = 2$ for infinitely many n , then $b_n = 2$ for all $n \geq N$.*

Proof. Suppose $b_n = b_{n+k} = 2$ and $b_{n+m} \neq 2$ for $0 < m < k$, where $n \geq N$ and $k > 1$. As before, this means $g_{n,k}(\frac{1}{2}) = 0$, which expands to

$$\frac{1}{2} - \frac{1}{2} r_{n+1} = 2^{-k} + \sum_{m=2}^{k-1} r_{n+m} 2^{-m}.$$

If $r_{n+1} = -1$, the absolute value of the left-hand side is 1, while the absolute value of the right-hand side is less than 1, so equality cannot hold. If $r_{n+1} = 1$, the left-hand side is 0, but the largest-magnitude term on the right (2^{-k} if $k = 2$, or $r_{n+2} \cdot 2^{-2}$ if $k > 2$) has greater absolute value than all the other terms combined, so the right-hand side cannot be 0, and equality cannot hold. Hence if $b_n = 2$ for infinitely many n , then $b_n = 2$ for all $n \geq N$. \square

Now suppose $b_n = 2$ does not hold for infinitely many n . Then -1 and 1 must be the only values taken by b_n for $n \geq N$. If $r_n = r_{n+1}$, we get $g_{n,1}(r_n) = 0 \Rightarrow r_n - 1 + r_n = 0 \Rightarrow r_n = \frac{1}{2}$. Hence neither -1 nor 1 can occur twice in a row in the sequence (r_n) . So b_n must alternate between 1 and -1 for $n \geq N$. Therefore for some $n \geq N$ we must have $r_n = -1, r_{n+1} = 1, r_{n+2} = -1$. This means $g_{n,2}(-1) = 0$, that is,

$$(-1)^2 - 1 + (-1) + 1 \cdot (-1) = 0.$$

This does not hold, giving a contradiction. Hence $r_n = \frac{1}{2}$ must hold for infinitely many n , and therefore for all $n \geq N$. We are done. \square