HÖJDPUNKTEN 2026

Gymnasie Competition, 20 March 2026

Solutions

Problem 1. A scatterbrained time traveller got stuck in the year 1 A.D with only a broken time machine available. The time machine only has three working buttons, which do the following:

- $(+1)$ — Travel a year forward.
- (-1) — Travel a year backwards.
- $(\times 3)$ — Triple the current year.

The time machine only has energy left for ten button presses. Describe how the time traveller can make it back to the year 2026 A.D. *(Only answer required)*

Solution. **Solution**

```
(x3) -> 3 A.D
(x3) -> 9 A.D
(-1) -> 8 A.D
(x3) -> 24 A.D
(+1) -> 25 A.D
(x3) -> 75 A.D
(x3) -> 225 A.D
(x3) -> 675 A.D
(x3) -> 2025 A.D
(+1) -> 2026 A.D
```

Alternatively: to get from 9 to 25 one can also press `x3-1-1`.

□

Problem 2. John has two lawn mowers named Alfons and Bosse that consume diesel at rates of $8\frac{\text{dL}}{\text{h}}$ and $6\frac{\text{dL}}{\text{h}}$, respectively. To compare the two, he first uses Alfons to cut half the lawn, and then Bosse to cut the other half. In total this took 15 minutes and consumed 1.7 dL of diesel. Which lawn mower consumed the least fuel?

Solution. Let x be the time (in minutes) for which Alfons was used. Then $15 - x$ is the total time Bosse was used. Computing the amount of diesel they must have consumed gives

$$\begin{aligned} x\,\text{min} \cdot \frac{8\frac{\text{dL}}{\text{h}}}{60\frac{\text{min}}{\text{h}}} + (15-x)\,\text{min} \cdot \frac{6\frac{\text{dL}}{\text{h}}}{60\frac{\text{min}}{\text{h}}} &= 1.7\,\text{dL} \\ x \cdot \frac{8}{60} + (15-x) \cdot \frac{6}{60} &= 1.7 \\ 8x + (15-x) \cdot 6 &= 60 \cdot 1.7 \\ x \cdot (8-6) &= 60 \cdot 1.7 - 15 \cdot 6 \\ x &= (60 \cdot 1.7 - 15 \cdot 6)/2 \\ &= (102 - 90)/2 \\ &= 6. \end{aligned}$$

Therefore Alfons used $6 \cdot 8/60 = 0.8$ dL of diesel in total and Bosse used 0.9 dL in total. Since they each cut the same amount of lawn, Alfons is the more fuel-efficient mower. □

Problem 3. A grocery store has a sale: Buy at least four fruits and get the cheapest one for free. The first customer buys three oranges and one banana. The second one buys three oranges and two lemons. The third one buys two bananas and two lemons. It turns out that all three customers paid exactly 35 kr for their fruits. Fins the price for each individual fruit.

Solution. Let a, b and c denote the unit prices of oranges, bananas and lemons, respectively. If $c \geq b$, customer 2 would have paid more than customer 1, which they did not, so we must have $b > c$. If $a \geq b$, customer 2 would have paid more than customer 3, so we must have $b > a$. We conclude that the bananas are the most expensive of the three fruits. It remains to determine whether a or c is larger. Customers 1 and 3 pay $2a + b$ and $b + 2c$ respectively for their fruit. If $c > a$, then

$$35 = 2a + b < 2c + b < b + 2c = 35,$$

which is impossible. Hence $a \geq c$. We have now determined the price ordering of the fruits, so we know how many fruits of each kind each customer must pay for. This gives the system of equations

$$\begin{cases} 2a + b = 35 & (1) \\ 3a + c = 35 & (2) \\ 2b + c = 35 & (3) \end{cases}$$

From (1) and (2) we get $b = 35 - 2a$ and $c = 35 - 3a$. Substituting into (3) gives

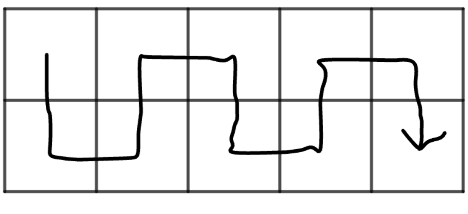
$$\begin{aligned} 2(35 - 2a) + (35 - 3a) = 35 \\ \iff 105 - 7a = 35 \\ \iff a = \frac{105 - 35}{7} = 10. \end{aligned}$$

Hence $b = 35 - 20 = 15$ and $c = 35 - 30 = 5$.

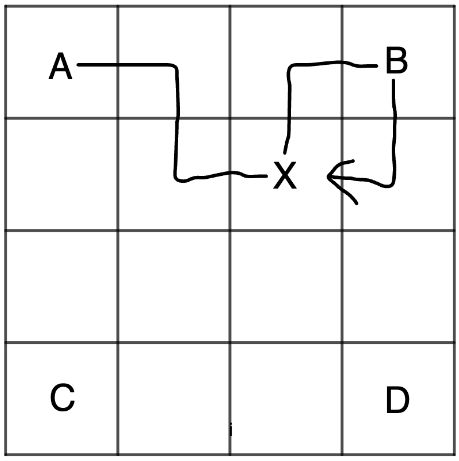
Answer: The oranges cost 10 kr, the bananas cost 15 kr and the lemons cost 5 kr. □

Problem 4. Styrbjörn walks around in a grid of size $n \times m$, where $n \geq 2$ and $m \geq 2$ are integers. He starts in one of the corners, and every minute he walks into a square next to (i.e. sharing a side with) the one he is in. Styrbjörn is a bit restless, and therefore he does not want to visit the same square twice. He also does not want to walk two steps in a row in the same direction. For which values of n and m can Styrbjörn visit all $n \cdot m$ squares?

Solution. He can visit all squares if and only if either n or m equals 2. If n or m equals 2, Styrbjörn can pass through every square by zig-zagging through the grid. See picture:

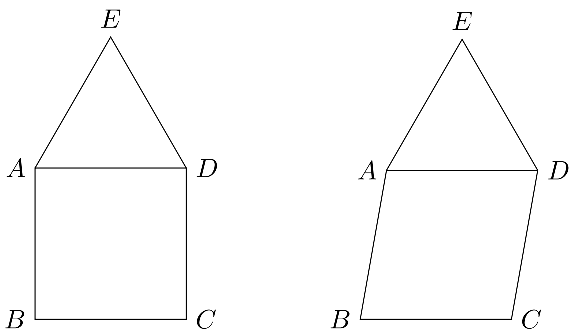


Now suppose neither n nor m equals 2. Let A be the corner where Styrbjörn starts, and let B, C, and D be the other corners. Note that two steps always take Styrbjörn to a square that is one diagonal step away from the one he was on. There is only one square that is one diagonal step away from B; call it X. To get to B, Styrbjörn must first pass through X (since $m, n \geq 3$, Styrbjörn cannot go directly from A to B). Likewise, if he is to take two steps from B he would have to return to X, which he is not allowed to do. So he must finish either at B or on the move after B. The same must hold for C and D, but this is impossible since Styrbjörn can only end his walk once. Hence Styrbjörn cannot visit all squares.



□

Problem 5. The picture shows a two dimensional model of a house consisting of six line segments of equal length. A strong gust of wind has made it so that the walls of the house (AB and CD) have started to tilt (by an angle smaller than 30°). Prove that the angle $\angle BEC$ does not depend on the tilt and calculate the size of this angle.



Solution. Let $x = \angle AEB$, $y = \angle BEC$ and $z = \angle CED$. The triangle $\triangle ADE$ is equilateral, so

$$x + y + z = \angle AED = 60°. \quad (*)$$

We also have

$$x = \angle ABE = \angle ABC - \angle EBC,$$

$$z = \angle DCE = \angle DCB - \angle ECB.$$

From the angle sum in triangle $\triangle BCE$ we get

$$y = 180^\circ - \angle EBC - \angle ECB,$$

hence

$$x + z - y = \angle ABC + \angle DCB - 180^\circ = 0 \quad (**)$$

where we used the fact that the sum of two adjacent angles in a rhombus ($\square ABCD$) is 180°. Finally, subtracting $(**)$ from $(*)$ gives

$$2y = 60^\circ,$$

i.e. $y = 30^\circ$. □

Problem 6. In the old castle, the royal glazier works with creating a new window for the throne room. She has to place ten purple panes and six yellow panes in a 4×4 grid such that no two rows and no two columns have the same number of yellow panes. In how many ways can she do this?

Solution. If any row or column has zero purple squares, the remaining rows can contain at most $2 + 3 + 4 \leq 9$ purple squares. Hence every row and column must contain at least one purple square. We will show that the colouring is uniquely determined by the number of purple squares in each row and column. This reduces the question to counting how many ways the row counts and column counts can be permuted. Both the row counts and the column counts can be chosen in $4! = 24$ ways, so by the multiplication principle the total number of ways to place the glass pieces is $(4!)^2 = 576$.

Let R_1, R_2, R_3, R_4, C_1, C_2, C_3, C_4 denote the rows and columns that are to contain 1, 2, 3 and 4 purple squares, respectively. First of all, R_4 and C_4 must be filled with purple squares. Then R_1 and C_1 each get one coloured square, which means the remaining squares in R_1 and C_1 must be yellow. Consequently in R_3 and C_3 all squares not in C_1 or R_1 are forced to be purple. What remains is the cell at the intersection of C_2 and R_2, where the last yellow square is placed. Now ten squares have been coloured purple and six yellow in a way that satisfies all conditions, and the process shows that the colouring is unique.

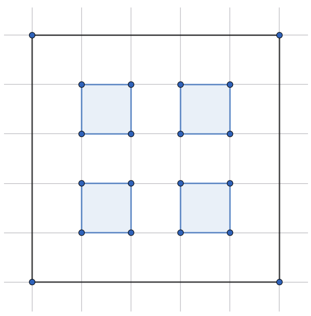
Answer: $(4!)^2 = 576$ ways. □

Problem 7. A square carpet is sewn together from square pieces of fabric of which half are large and half are small. The large pieces of fabric are 2×2 dm and the small ones are 1×1 dm. The pieces are sewn together without any overlap. What is the smallest possible size of this carpet?

Solution. Let n be the number of pieces of each kind and s the side length of the carpet. Then

$$5n = s^2,$$

so s is divisible by 5. We first check whether $s = 5$ works. This gives $n = 5^2/5 = 5$, but if we try to place the large pieces on a 5×5 dm carpet we see that each large piece must cover exactly one of the squares in the picture.



There are only four such squares, so we can fit at most 4 large pieces, which is not enough. Hence $s = 5$ does not work.

The next number to check is $s = 10$, and we see that it works. Divide a 10×10 grid into 25 large squares. Choose five of these squares and divide each of them into four small squares. This gives 20 small squares and 20 large squares, as desired.

Answer: The carpet is at least 1×1 square metre. □

Problem 8. Find all solutions to the system of equations

$$\begin{cases} p + q + r = s \\ p + 2q + 3r = 5t \end{cases}$$

where p, q, r, s and t are prime numbers.

Solution. Subtracting the first equation from the second, we get

$$q + 2r = 5t - s.$$

Both s and t are obviously greater than 2, hence odd, so the right-hand side is odd − odd = even. This means q is even, i.e. $q = 2$.

From the first equation we now see that $p + r$ must be odd. This gives two cases:

Case 1 $p = 2$ and r odd: The second equation becomes

$$2 + 4 + 3r = 5t.$$

But the left-hand side is a multiple of 3, so $t = 3$, and hence $r = (15 - 4 - 2)/3 = 3$. The first equation then gives $s = 7$. Substituting back, we see that this solves the system.

Case 2 p odd and $r = 2$: The second equation now becomes

$$p + 4 + 6 = 5t \iff p = 5t - 10.$$

The right-hand side is now a multiple of 5, so $p = 5$ and $t = 3$. But substituting this into the first equation gives $s = 5 + 2 + 2 = 9$, which is not prime.

Answer: The system has only one solution, namely $(p, q, r, s, t) = (2, 2, 3, 7, 3)$. □

Problem 9. Consider $n \geq 3$ points in the plane where no three lie on a line. Choose one of the points and draw two lines from it to two other points. This creates an angle (we choose the one smaller than 180°). What is the result if we sum all such angles?

Solution. Given any three points, the angles formed between them are the interior angles of the triangle they form, and so sum to 180°. Hence the total is 180° times the number of ways to choose three points out of n, giving the answer $\binom{n}{3} \cdot 180°$. □

Problem 10. Let $\triangle ABC$ be a right-angled triangle with $\angle ABC = 90°$ and $|AB| < |BC|$. Let D be a point on the hypotenuse AC such that $|AB| = |BD|$. The point T lies on side BC and is such that $\angle ATB = \angle CTD$. Prove that the line through D perpendicular to BD splits the segment CT in half.

Solution. Let D' be the reflection of D across BC. Then $\angle CTD' = \angle CTD = \angle ATB$, so the points A, T and D' lie on a line.

Set $\angle BAD = \angle ADB = x$, which further gives

$$\begin{aligned}\angle ABD &= 180^\circ - 2x\\ \angle DBC &= 90^\circ - \angle ABD = 2x - 90^\circ\\ \angle D'BC &= \angle DBC = 2x - 90^\circ\\ \angle ABD' &= 90^\circ + \angle D'BC = 2x.\end{aligned}$$

Since $|BD'| = |BD| = |AB|$, the triangle $\triangle ABD'$ is isosceles, so

$$\angle BAD' = \angle BD'A = 90^\circ - \frac{1}{2}\angle ABD' = 90^\circ - x.$$

By symmetry we therefore have $\angle BDT = \angle BD'T = \angle BD'A$, which gives

$$\angle TDC = 180^\circ - \angle ADB - \angle BDT = 180^\circ - x - (90^\circ - x) = 90^\circ,$$

so the angle $\angle TDC$ is right.

Let M be the midpoint of CT. By **Thales' theorem**, $DM = TM$, so

$$\angle TDM = \angle DTM = \angle DTC = \angle ATB = 90^\circ - \angle BAT = 90^\circ - \angle BAD' = x.$$

As noted earlier, $\angle BDT = \angle BD'A = 90^\circ - x$, which gives

$$\angle BDM = \angle BDT + \angle TDM = 90^\circ - x + x = 90^\circ,$$

which is what we wanted to show.

One can also show and use that:

- $(ABTD)$ is a cyclic quadrilateral.
- B is the centre of the circle (ADD').
- $(TDD'C)$ is a cyclic quadrilateral and is tangent to BD.

□

Problem 11. Find all functions f, from the real numbers to the real numbers, such that the equality

$$f(f(x) + f(y) + yf(f(x))) = yf(f(x)) + xf(x)$$

holds for all real numbers x and y.

Solution.

$$\text{Let } P(x,y): \quad f\big(f(x) + f(y) + yf(f(x))\big) = yf(f(x)) + xf(x).$$

We show that the only solution is $f \equiv 0$.

Set

$$a = f(0).$$

From $P(x,0)$ we get

$$f\big(f(x) + a\big) = xf(x). \tag{1}$$

From $P(0,0)$ we get

$$f(2a) = 0. \tag{2}$$

Now set $x = 2a$ in (1). We obtain

$$f\big(f(2a) + a\big) = 2a\, f(2a).$$

By (2) this becomes

$$f(a) = 0. \tag{3}$$

From $P(0, y)$ we get

$$f\big(a + f(y) + yf(f(0))\big) = yf(f(0)).$$

Since $f(0) = a$ and $f(a) = 0$ by (3), it follows that

$$f(a + f(y)) = 0 \qquad \forall y \in \mathbb{R}. \tag{4}$$

Now (1) and (4) have the same left-hand side, hence

$$xf(x) = 0 \qquad \forall x \in \mathbb{R}.$$

Thus

$$f(x) = 0 \qquad \forall x \neq 0. \tag{5}$$

It remains to determine $f(0)$. By (5), in particular $f(1) = 0$. Now set $(x, y) = (1, 1)$ in P:

$$f\big(f(1) + f(1) + f(f(1))\big) = f(f(1)) + f(1).$$

Since $f(1) = 0$, this becomes

$$f(f(0)) = f(0),$$

that is,

$$f(a) = a.$$

Combined with (3) this yields

$$a = 0.$$

So $f(0) = 0$ as well.

From this together with (5) we conclude

$$f(x) = 0 \qquad \forall x \in \mathbb{R}.$$

Hence the only function is

$$\boxed{f(x) = 0 \text{ for all } x \in \mathbb{R}.}$$

□

Problem 12. Let a be a positive integer. Let a_1, a_2, a_3, $\ldots$, be the infinite sequence of positive integers satisfying $a_1 = a$ and $a_{n+1} = a_n + \gcd(a_n, n)$ for all positive integers n.

(a) Find all values of a for which $a_n \leq n + 2026$ for all positive integers n.

(b) For each positive integer a, prove that there exists a constant C such that $a_n \leq Cn$ for all positive integers n.

[*The greatest common divisor,* $\gcd(x, y)$*, of two integers* x *and* y *is the largest positive integer thet divides both* x *and* y*.*]

Solution. (a) We claim that the only possible values are $a = 1, 2$.

If $a = 1$, then $a_2 = 2$ and $a_3 = 4$. Then for every $n \geq 3$, if $a_n = n + 1$ we have

$$a_{n+1} = a_n + \gcd(a_n, n) = (n + 1) + \gcd(n + 1, n) = n + 2.$$

Hence $a_n = n + 1$ for all $n \geq 3$.

If $a = 2$, then $a_2 = 3$, and the same induction gives

$$a_n = n + 1 \qquad \text{for all } n \geq 2.$$

In both cases, $a_n \leq n + 2026$ for all n.

Now suppose $a > 2$. Then

$$a_2 = a + \gcd(a, 1) = a + 1 \geq 4 = 2 \cdot 2.$$

We show by induction that $a_n \geq 2n$ for all $n \geq 2$. Suppose $a_n \geq 2n$.

If $a_n = 2n$, then

$$a_{n+1} = a_n + \gcd(a_n, n) = 2n + n = 3n \geq 2(n + 1).$$

If $a_n > 2n$, then since a_n is an integer, $a_n \geq 2n + 1$, and so

$$a_{n+1} \geq a_n + 1 \geq 2n + 2 = 2(n + 1).$$

Thus $a_n \geq 2n$ for all $n \geq 2$. Setting $n = 2027$ gives

$$a_{2027} \geq 4054 > 4053 = 2027 + 2026,$$

so the condition cannot hold. Consequently the only possible values are

$$\boxed{1, 2}.$$

(b) For $n \geq 2$, set

$$b_n = \left\lceil \frac{a_n}{n - 1} \right\rceil.$$

We show that $b_{n+1} \leq b_n$ for all $n \geq 2$.

Set $k = b_n$. Then we may write

$$a_n = k(n - 1) - d$$

for some integer $d \geq 0$. Then

$$\begin{aligned} a_{n+1} &= a_n + \gcd(a_n, n) \\ &= k(n - 1) - d + \gcd(k(n - 1) - d, n) \\ &= kn - (k + d) + \gcd(k + d, n) \leq kn, \end{aligned}$$

since $\gcd(k + d, n) \leq k + d$. Hence

$$\left\lceil \frac{a_{n+1}}{n} \right\rceil \leq k = \left\lceil \frac{a_n}{n - 1} \right\rceil,$$

so (b_n) is non-increasing. Therefore

$$b_n \leq b_2 = \left\lceil \frac{a_2}{1} \right\rceil = a_2 = a + 1.$$

For $n \geq 2$ we then have

$$a_n \leq b_n(n - 1) \leq (a + 1)(n - 1) \leq (a + 1)n.$$

Furthermore $a_1 = a \leq a + 1$. So

$$\boxed{C = a + 1}$$

works.

□