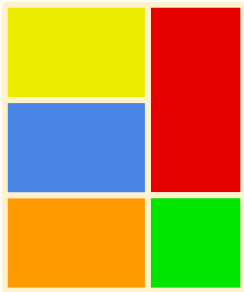
Open Competition 2025

Solutions

Problem 1. A rectangle is partitioned into n smaller rectangles which are to be painted in different colors. In preparation, strips of tape need to be placed covering all of the edges of the rectangles such that no two strips of tape cross. What is the fewest number of strips of tape needed?



Solution. We claim that the minimal number of tape pieces required is $n+3$.

Consider an arrangement of tape which covers all of the edges of the rectangles and which uses as few tape pieces as possible. For such an arrangement, each end of a tape piece will lie in a three-way intersection, a four-way intersection or in one of the large rectangle's four corners.

By each three- and four-way intersection there are exactly twice as many rectangle corners as tape piece ends present. Moreover, by each one of the large rectangle's four corners there are exactly two tape piece ends present.

Since there are exactly $4n$ rectangle corners in total, and each tape piece has two ends, an arrangement of tape using as few tape pieces as possible will always use exactly

$$\frac{1}{2}\left(\frac{1}{2}(4n-4)+2\cdot 4\right)=n+3$$

tape pieces, which was what we wanted to show.

Problem 2. Let n and k be positive integers with $n \geq \max\{k, 3\}$. Alice and Bob play a game on an (undirected and simple) graph G. At the start of the game, G is a complete graph on n nodes. Each round, Alice first selects k edges in G, and then Bob removes one of the edges Alice had selected.

The game ends when G only has $n-1$ edges left. Then Alice wins if G is connected and otherwise Bob wins. What is the largest positive integer k, in terms of n, for which Alice wins if both players play optimally?

Solution. We claim that the largest positive integer k for which Alice always wins is $k = 3$, given that both players play optimally.

Lemma 1. Alice wins if $k \leq 3$.

Proof of Lemma 1. It is well-known that an (undirected and simple) graph on n nodes with more than $n - 1$ edges contains at least one cycle.

Thus, on her turn, Alice can always choose $k \leq 3$ edges from the same cycle c in G. When Bob removes one of these edges, e, the resulting graph will still be connected. This holds since if two nodes are connected by a path before e was removed it will either be the case that (1) the same path still connects them after e is removed, or (2) if e was part of the path then e can be replaced by the remaining edges in c to form a new path connecting the nodes.

Therefore Alice can guarantee her victory when $k \leq 3$. □

Lemma 2. Bob wins if $k > 3$.

Proof of Lemma 2. Let Bob choose a triangle T in G at the start of the game. This is possible because G is a complete graph on n nodes at that time.

On his turn, Bob can always preserve T by removing one of the at least $k - 3 \geq 1$ many edges picked out by Alice which are not part of T.

When there are only $n - 1$ edges left in the graph, the triangle T will still be preserved. But a graph on n nodes and $n - 1$ edges cannot be connected if it contains a cycle, in this case T.

Therefore Bob can guarantee his victory when $k > 3$. □

Hence, Bob wins when $k > 3$ and Alice wins when $k \leq 3$. Thus, $k = 3$ is the largest possible k for which Alice wins.

Problem 3. A frog is at the point $(0,0)$ in the plane and starts jumping. Its first jump has length 1, and each subsequent jump is twice as long as the previous one. Every jump is made parallel to one of the coordinate axes. Which points can the frog reach by jumping in this way?

Solution. Notice that the frog is only able to reach points whose integer coordinates (x,y) satisfy that $x+y$ is odd. We claim that the frog can reach all of these points.

Lemma 1. Let $n \in \mathbb{N}$. The frog can jump to all points (x,y) with integer coordinates such that $x+y$ is odd and $|x|+|y| \leq 2^n - 1$ in exactly n jumps.

Proof of Lemma 1. We will prove the lemma by induction on n.

Base case $(n=1)$: On its first jump, the frog can jump to $(1,0)$, $(0,1)$, $(-1,0)$ or $(0,-1)$, which encompasses all points (x,y) with integer coordinates such that $x+y$ is odd and $|x|+|y| \leq 2^1 - 1$.

Induction step $(n>1)$: Notice that the frog can jump to all points (x',y') with integer coordinates such that $x'+y'$ is odd and $|x'|+|y'| \leq 2^{n-1}-1$ using exactly $n-1$ jumps. Consider a point (x,y) with integer coefficients such that $x+y$ is odd and $|x|+|y| \leq 2^n - 1$. We have $x \neq y$ since $x+y$ is odd. By symmetry, we can assume that $x > y \geq 0$. The point $(x-2^{n-1},y)$ is then odd since $n>1$. We have

$$\begin{cases} x \geq 2^{n-1} \Rightarrow |x-2^{n-1}|+|y| = x-2^{n-1}+y \leq 2^n - 1 - 2^{n-1} = 2^{n-1}-1 \\ x < 2^{n-1} \Rightarrow |x-2^{n-1}|+|y| = 2^{n-1}-x+y = 2^{n-1}-(x-y) \leq 2^{n-1}-1. \end{cases}$$

By the induction hypothesis, the frog can reach $(x-2^{n-1},y)$ after exactly $n-1$ moves and can thus reach (x,y) after exatcly n jumps (the n:th jump is of length 2^{n-1}). Thus, we have shown that the frog can jump to all points (x,y) with integer coordinates such that $x+y$ is odd and $|x|+|y| \leq 2^n - 1$ using exactly n jumps, which implies the induction step. □

Therefore the set of points to which the frog can jump to is exactly the set of points (x,y) with integer coordinates such that $x+y$ is odd.

Problem 4. A stick of length 1 is split between a countable number of people $P_1, P_2, \ldots$. First, P_1 chooses a number U_1 uniformly from $[0, 1]$ and the stick is split in a piece of length U_1 and a piece of length $1 - U_1$. P_1 keeps the stick of length U_1 and passes to P_2 the remaining piece. P_2 then chooses U_2 uniformly from $[0, 1]$, and splits the stick into two pieces of length $U_2(1 - U_1)$ and $(1 - U_2)(1 - U_1)$. The first piece is kept by P_2 and the other stick is given to P_3, and the process continues so that P_n recieves a stick of length $U_n(1 - U_{n-1}) \ldots (1 - U_1)$. What is the probability that the stick that every person recieves is shorter than $1/2$?

Solution. The answer is $1 - \ln 2$.

The function $f\colon [0, 1] \to [0, 1]$ is defined to be such that $f(x)$ is the sought-after probability for the corresponding problem where the stick has length x instead. If $x < \frac{1}{2}$, then no piece can be longer than $\frac{1}{2}$ and if $x > \frac{1}{2}$ we can use the law of total probability for U_1. Therefore, we get that

$$f(x) = \begin{cases} 1, & x < \frac{1}{2} \\ \frac{1}{x} \int_{x-\frac{1}{2}}^{x} f(t)\, dt, & x \geq \frac{1}{2} \end{cases}.$$

For $x \geq \frac{1}{2}$, after multiplication by x and differentiating, we get that

$$f(x) + xf'(x) = f(x) - f(x - \tfrac{1}{2}) \implies xf'(x) = -1$$

so $f(x) = c - \ln x$ for $x \geq \frac{1}{2}$. The constant c is determined by $1 = f(\frac{1}{2}) = c + \ln 2$, so $f(x) = 1 - \ln 2 - \ln x$. The sought after value is $f(1) = 1 - \ln 2$.

Comment: The function $f(x)$ above is a rescaled version of the so-called "Dickmann function" $\rho(x) = f(2x)$ which appears in many different situations. For example, if n is chosen uniformly at random from the integers between 1 and N for some large integer N, then $\rho(x)$ approximates the probability that $\rho_{\max}(n) < n^{\frac{1}{x}}$, where $\rho_{\max}(n)$ denotes the largest prime factor of n. For $1 < x < 2$ the relation $\rho(x) = 1 - \ln x$ holds, but for $x > 2$ there is no general closed form for ρ.

Problem 5. Lisa the astronaut is constructing humanity's first vegetable plantation on planet Mars. She is building it out of a grid of hexagonal plantation modules whose side lengths are all 1 meter long. Right above the middle of the middle plantation module, she places a light source that illuminates all modules within a circle of radius 100m. For a plantation module to work properly, it must be completely illuminated. Given that Lisa has filled the illuminated circle with as many plantation modules as possible, determine the perimeter of the plantation.

Solution. Place the light source at the origin and let u, v and w be three vectors which point to three pairwise non-adjacent corners of the centermost hexagon. Notice that every hexagon in the grid has its center at a point which can be expressed as $au + bv + cw$ where a, b, c are integers with $3 \mid a + b + c$. In particular, there are six hexagons with centers at $\pm 99u$, $\pm 99v$ and $\pm 99w$ and these hexagons are tangent to the illuminated circle in the points $\pm 100u$, $\pm 100v$ and $\pm 100w$, which by symmetry partition the circumference of the plantation into six congruent "paths". All these "paths" have the same length, so it suffices to compute the length of one of them and then multiply the result by 6.

Consider the path from $A := 100u$ to $B := -100v$ whose total displacement is $100w$. This path will consist of unit length steps, each one in one of the directions $\pm u$, $\pm v$ or $\pm w$. However, notice that the only steps that may appear in the path are steps which takes the path closer to B, i.e. w-steps, $(-u)$-steps and $(-v)$-steps. Moreover, each w-step must either be followed by a $(-u)$-step or a $(-v)$-step and in the same way every $(-u)$-step or $(-v)$-step must be followed by a w-step. Therefore, every other step is a w-step and every other step is a $(-u)$-step or a $(-v)$-step. Each $(-u)$-step and $(-v)$-step takes the path one half w-step in the w-direction, so for the path to consist of a total displacement of 100 steps in the w-direction we must have that it consists of 67 w-steps and 66 ($-u$ or $-v$)-steps. Therefore, the total number of steps around the plantation is $6 \cdot (66 + 67) = 798$.
Answer: 798m

Problem 6. Let ABC be a scalene triangle with incenter I and circumcircle Ω. Let M be the midpoint of arc BC on Ω containing A. Let D be the intersection of lines BC and AM. Let J be the second intersection of line DI with the circumcircle of triangle BIC. Prove that the tangent to the circumcircle of triangle BIC at J bisects segment DM.

Solution. Let L be the midpoint of segment DM and denote by ω the circumcircle of triangle BIC. We show that the line LJ is tangent to ω.

From Power of a point at D we get

$$DA \cdot DM \stackrel{\Omega}{=} DB \cdot DC \stackrel{\omega}{=} DI \cdot DJ$$

so M, A, I, and J lie on a circle, which we denote Γ. This now implies that

$$\angle MJD = \angle MJI = \angle MAI = 90^\circ$$

where the last equality follows from the incircle-excircle lemma, so L is the circumcenter of the right-angled triangle DML which gives that $LM = LJ = LD$. We now present two ways of finishing the proof:

Method 1: Let I_A be the A-excenter of triangle ABC. Recall that I_A lies on line AI and that I_AI is a diameter of ω. Since $\angle IJM = \angle I_AJI = 90^\circ$, the points M, J, I_A all lie on a line, which we denote by ℓ. Now

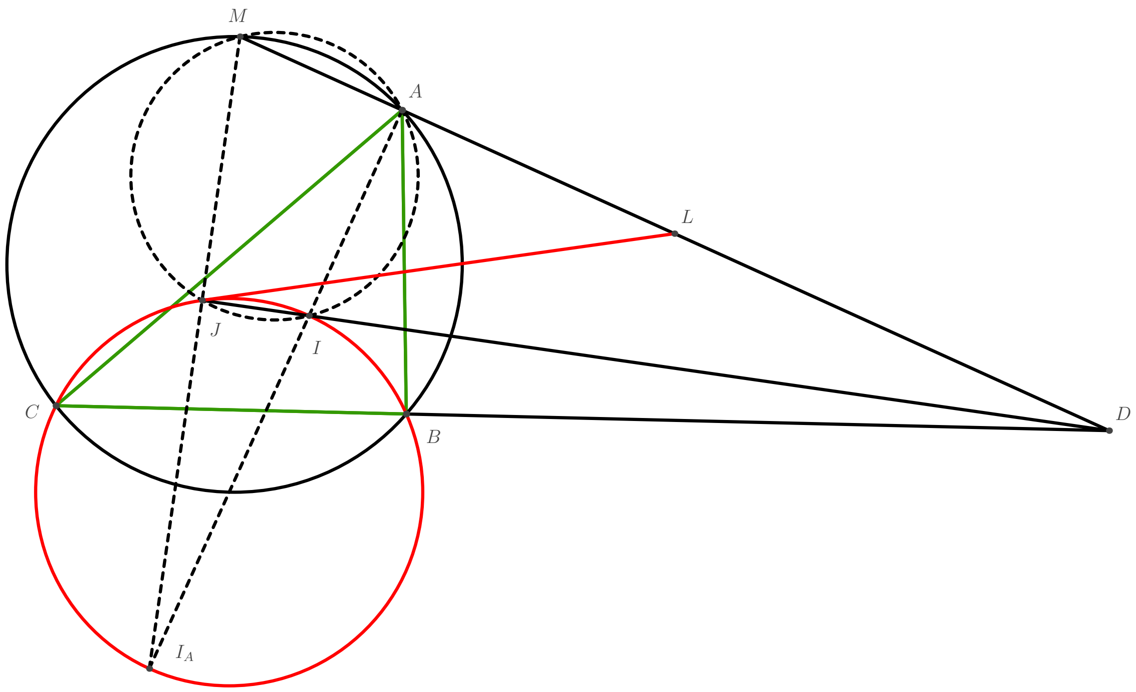
$$\angle I_AJL \stackrel{\ell}{=} \angle MJL = \angle LMJ = \angle AMJ \stackrel{\Gamma}{=} \angle AIJ = \angle I_AIJ$$

so LJ is tangent to ω by the tangent-secant theorem. This finishes the proof.

Method 2: Let N be the point on Ω diametrically opposite M - from the incircle-excircle lemma, N is the center of ω, so we wish to prove that $\angle NJL = 90^\circ$. The inscribed angle and tangent-secant theorems give that

$$\angle ALJ = 2\angle ADJ = 2\angle LDJ = 2\angle DJL = 2\angle ICJ = \angle INJ = \angle ANJ$$

so L, A, J, N are cyclic. But $\angle NAL = 90^\circ$ since N lies on the line AI, so $\angle NJL = 90^\circ$ as desired.



Problem 7. Let n be a positive integer. Oscar receives a bag containing n distinct positive integers and writes on a board all possible numbers of the form $xy + z$, where x, y and z are (not necessarily distinct) numbers from the bag. Given n, determine the minimum number of *distinct* numbers that Oscar may have written on the board.

Solution. For a set S of n positive integers, let

$$T(S) = \{xy + z \mid x, y, z \in S\}.$$

We claim that the smallest possible size of $T(S)$ is $n^2 + n - 1$.

Lemma 1. If $S = \{1, \dots, n\}$ then $|T(S)| = n^2 + n - 1$.

Proof of Lemma 1. If $S = \{1, \dots, n\}$ then it is clear that $T(S) = \{2, \dots, n^2 + n\}$. Thus we have that $|T(S)| = n^2 + n - 1$, which was what we wanted to prove. □

Lemma 2. For all sets S of n positive integers it holds that $|T(S)| \geq n^2 + n - 1$.

Proof of Lemma 2. Let e be the smallest element of S and let E be the largest element of s.

We claim that all numbers of the forms

(i) $xE + z$, where $x, z \in S$, and

(ii) $e^2 + z$, where $z \in S \backslash \{E\}$

are pairwise distinct. This claim can be shown by considering the following cases:

- If $x_1 E + z_1 = x_2 E + z_2$ we get that $z_1 = z_2$ by considering the equation modulo E. Thereafter we immediately get that $(x_1, z_1) = (x_2, z_2)$.
- If $e^2 + z_1 = e^2 + z_2$ we must have that $z_1 = z_2$.
- Notice that $e^2 + z_1 < e^2 + E \leq eE + e \leq xE + z_2$ for all $x, z_2 \in S$, $z_1 \in S \backslash \{E\}$.

Since there are n^2 numbers of the form (i) and and $n - 1$ numbers of the form (ii) in $T(S)$, it follows that $|T(S)| \geq n^2 + n - 1$ for all sets S of n positive integers. □

Therefore the minimum number of distinct numbers that Oscar may have written on the board is equal to $n^2 + n - 1$.

Problem 8. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ satisfying that

$$d\left(\sum_{i=1}^{N} a_i\right) \leq d\left(\sum_{i=1}^{N} f(a_i)\right)$$

for all N, a_1, ..., $a_N \in \mathbb{N}$. (Here, $d(n)$ denotes the number of divisors of n.)

Solution. We claim that the family of solutions $f(n) = cn$, where c is any constant positive integer, constitutes the set of possible solutions to the given functional equation. It is obvious that all these functions satisfy the functional equation.

Lemma 1. If x and y are positive integers such that

$$d(Nx) \leq d(Ny)$$

for all $N \in \mathbb{N}$, then $x \mid y$.

Proof of Lemma 1. Let x and y be positive integers with $x \nmid y$. We will show that there exists some $N \in \mathbb{N}$ for which $d(Nx) > d(Ny)$.

Because $x \nmid y$ there exists a prime p such that $\nu_p(x) > \nu_p(y)$.

Consider $N = y^E p^{-E\nu_p(y)}$ for any positive integer E. Then

$$\frac{d(Nx)}{d(Ny)} \geq \frac{\nu_p(x)+1}{\nu_p(y)+1} \prod_{\substack{q \mid y \\ q \neq p}} \frac{(E\nu_q(y) + \nu_q(x) + 1)}{((E+1)\nu_q(y) + 1)}.$$

Since the right-hand side tends to $\frac{\nu_p(x)+1}{\nu_p(y)+1} > 1$ as E approaches ∞, there exists some E for which the right-hand side is larger than 1. Hence, we get that

$$\frac{d(Nx)}{d(Ny)} > 1$$

which was what we wanted to show. □

By applying Lemma 1 to the functional equation we get that

$$\sum_{k=1}^{N} a_k \;\Bigg|\; \sum_{k=1}^{N} f(a_k)$$

for alla $N, a_1, \ldots, a_N \in \mathbb{N}$.

Therefore, we must have $n \mid f(n)$ for each $n \in \mathbb{N}$. Let $g(n) = \frac{f(n)}{n}$. Hence, for all $A, n \in \mathbb{N}$, we have that

$$\begin{aligned} An + 1 &\mid Af(n) + f(1) \\ An + 1 &\mid An \cdot g(n) + g(1) \\ An + 1 &\mid -g(n) + g(1) \end{aligned}$$

so $g(1) - g(n)$ has infinitely many divisors and must therefore equal 0.

Thus, all solutions to the functional equation satisfy that $f(n) = cn$ for all $n \in \mathbb{N}$, for some fixed $c \in \mathbb{N}$, as desired.

Problem 9. Let a_1, a_2, a_3, $\dots$, be an infinite sequence of distinct positive integers, and let N be a positive integer. Suppose that, for each integer $n > N$, a_n is equal to the smallest positive integer which cannot be written as a sum of distinct elements of $\{a_1, \dots, a_{n-1}\}$.

Prove that there exists a positive integer M such that $a_m = 2a_{m-1}$ for all $m > M$.

Solution 1. Let $S_m = a_1 + ... + a_m$ and let $x_m = S_m + 1 - a_{m+1}$. Note that

- $x_m \geq 0$ since we definitely can't express $S_m + 1$ as a sum of the numbers from $\{a_1, ..., a_m\}$, and so $a_{m+1} \leq S_m + 1$
- $x_m - x_{m+1} = a_{m+2} - 2a_{m+1} \geq 0$, since you can write any number from 0 to $a_{m+1} - 1$ using only the numbers $a_1, ..., a_m$ and so you can write any number between a_{m+1} and $2a_{m+1} - 1$ by adding a_{m+1}, implying that a_{m+2} is at least $2a_{m+1}$

But the two observations above show that x_m is a non-increasing sequence of positive numbers, and hence it's eventually constant. And when it's constant, we must have $a_{m+2} = 2a_{m+1}$.

Solution 2. Let A_n denote the set $\{a_1, \dots, a_n\}$ and let $S_n = \sum_{k=1}^{n} a_k$ for each $n \in \mathbb{N}$. Furthermore, let χ_n denote the set of positive integers less than or equal to S_n which cannot be written as a sum of elements in the set A_n.

For each $n \geq N$, notice that

$$a_{n+1} = \begin{cases} S_n + 1 & \text{if } \chi_n = \emptyset \\ \min(\chi_n) & \text{otherwise} \end{cases}.$$

Notice that the sequence (a_n) is unbounded. Therefore we can choose a positive integer M such that $M > N$ and $a_M > S_N$.

Lemma 1. For each $n > N$ it holds that $a_{n+1} \geq 2a_n$.

Proof of Lemma 1. That a_n is equal to the least positive integer which cannot be written as a sum of elements in the set A_{n-1} implies that all positive integers in the set $\{1, \dots, a_n - 1\}$ can be written as a sum of elements in the set A_{n-1}. Therefore, all positive integers in the set

$$\{1, \dots, a_n - 1\} \cup \{a_n\} \cup \{a_n + 1, \dots, a_n + (a_n - 1)\}$$

can be written as a sum of elements in the set A_n.

Therefore we must have that $a_{n+1} > a_n + (a_n - 1) \implies a_{n+1} \geq 2a_n$ for each $n > N$. □

Lemma 2. For all positive integers b and n it holds that $b \in \chi_n \iff S_n - b \in \chi_n$.

Proof of Lemma 2. It suffices to prove that $b \notin \chi_n \implies S_n - b \notin \chi_n$ for all $b < S_n$.

If $b \notin \chi_n$, then b is equal to the sum of elements in a subset B of A_n. Thus, notice that $S_n - b$ is equal to the sum of elements in the subset $A_n \backslash B$ of A_n, and therefore $S_n - b \notin \chi_n$. □

Lemma 3. For each $m \geq M$ it holds that $\chi_m = \emptyset$ and $a_{m+1} = 2a_m$.

Proof of Lemma 3. From Lemma 1 it follows that

$$S_m = S_N + (a_{N+1} + \dots + a_m) < a_M + \left(\frac{a_{m+1}}{2^{m-N}} + \dots + \frac{a_{m+1}}{2^1}\right) < 2a_{m+1}.$$

Suppose that $\chi_m \neq \emptyset$, so $a_{m+1} = \min(\chi_m)$. Moreover, we have that $S_m - a_{m+1} \in \chi_m$ from Lemma 2. Combining this with the inequality above we deduce that

$$a_{m+1} = \min(\chi_m) \leq S_m - a_{m+1} < a_{m+1}$$

which is a contradiction.

Therefore we must have that $\chi_m = \emptyset$ for each $m \geq M$. Thus we get that

$$a_{m+1} = S_m + 1 = (S_{m-1} + a_m) + 1 = 2a_m$$

which was what we wanted to prove. □

From Lemma 3 we have that $a_m = 2a_{m-1}$ for all $m > M$, as desired.

Problem 10. Let P be a non-constant polynomial with integer coefficients and positive leading coefficient. For each positive integer n, prove that there exists a positive integer c such that $P(x)+c$ is prime for at least n different integers x.

Solution. We can replace $P(x)$ by $P(x-a)+b$ for whatever integers a and b we please and prove the problem for the resulting polynomial instead. Therefore, we may (without loss of generality) assume that $P(1)=0$ and that $P(x)$ is strictly increasing for $x>0$.

Consider the function

$$\tilde{Q}(m) = \max_{1\le i\le m} (P(i+1)-P(i))$$

for $m \in \mathbb{N}$.

Notice that $\tilde{Q}(m) > 0$ since $P(x)$ is strictly increasing for $x > 0$. Furthermore, notice that for all sufficiently large $m \in \mathbb{N}$ we will have that $\tilde{Q}(m) = P(m+1) - P(m)$.

For each $m \in \mathbb{N}$, consider the sets

$$S_m = \{P+1, \dots, P+\tilde{Q}(m)\}$$

and

$$L_m = \{p(k) \mid k \in \mathbb{N},\ p \in S_m\}.$$

Since $P(k+1)-P(k) \le \tilde{Q}(m)$ for each integer $1 \le k \le m$, it must be the case that every integer between $1 = P(1)+1$ and $P(m+1)+\tilde{Q}(m)$ will appear in L_m.

Consider what happens when $m \in \mathbb{N}$ tends towards ∞. From the Prime Number Theorem we get that there will be $\Theta\left(\frac{P(m+1)+\tilde{Q}(m)}{\log(P(m+1)+\tilde{Q}(m))}\right) = \Theta\left(\frac{m^d}{\log(m)}\right)$ primes between 1 and $P(m+1)+\tilde{Q}(m)$. Then L_m will contain at least $\Omega\left(\frac{m^d}{\log(m)}\right)$ primes. Notice that $|S_m| = \Theta(\tilde{Q}(m)) = \Theta(m^{d-1})$. By the pigeonhole principle we therefore get that there exists a polynomial $p \in S_m$ such that $p(k)$ is a prime for eat least $\Omega\left(\frac{m^d}{\log(m)}\middle/ m^{d-1}\right) = \Omega\left(\frac{m}{\log(m)}\right)$ different integers k.

Thus, for all sufficiently large $m \in \mathbb{N}$, there will exist some polynomial $p \in S_m$ such that $p(k)$ is a prime for at least n different positive integers k, which was what we wanted to show.

Problem 11. Let ABC be a triangle with incenter I and circumcircle Ω. Let the reflection of line BC over line AI intersect Ω at points P and Q. Prove that the circumcenter of triangle PIQ lies on Ω.

Solution 1. Let I_A be the A-excenter of triangle ABC, let M be the second intersection of line AI with Ω, and let D be the intersection between line AI and side BC. Let O be the circumcenter of triangle ABC and let O' be the circumcenter of triangle PIQ. Let ω denote the circumcircle of triangle BIC and let Γ denote the circumcircle of triangle PIQ. From the incircle-excircle lemma, I_A lies on ω and M is the circumcenter of ω.

If $AB = AC$, then $O' = M$ clearly lies on Ω. Moving forward we may therefore assume that $AB \neq AC$, which implies that points A, B, C, M, D, P, I, Q and I_A are pairwise distinct.

From Power of a point in D we see that

$$DP \cdot DQ \overset{\Omega}{=} DB \cdot DC \overset{\omega}{=} DI \cdot DI_A$$

so I_A lies on Γ.

Let B' be the reflection of B over line AI. Note that B' lies on ω since the circumcenter M of ω lies on line AI. Moreover, B' lies on the circumcircle of triangle CMD, since

$$\angle DMC = \angle AMC = \angle ABC = \angle ABD = \angle DB'A = \angle DB'C.$$

Thus, Power of a point in A gives

$$AD \cdot AM \overset{(CMD)}{=} AC \cdot AB' \overset{\omega}{=} AI \cdot AI_A.$$

Since line AO is the reflection of line AI in the height from vertex A in triangle ABC, we have that $AO \perp PQ$. Together with $OP = OQ$, we get that AO is the perpendicular bisector of the segment PQ, so that $AP = AQ$.

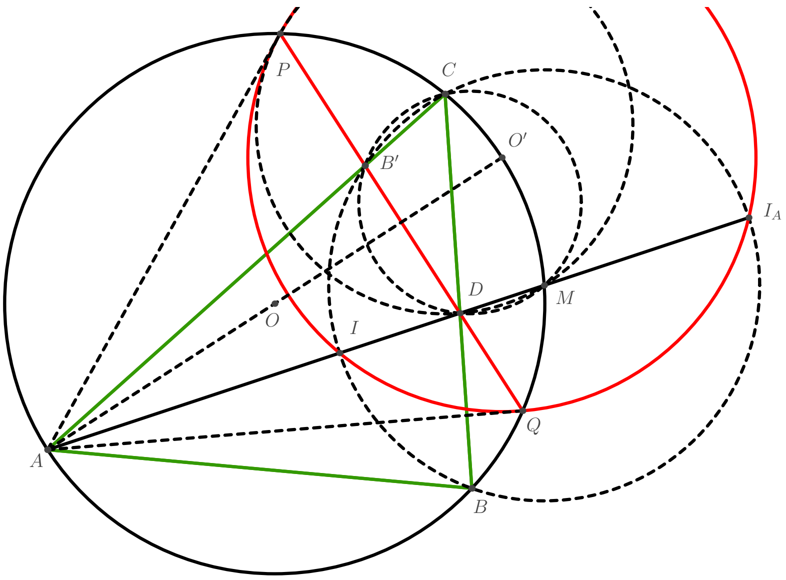
Now

$$\angle DMP = \angle AMP = \angle AQP = \angle QPA = \angle DPA$$

which implies that line AP is tangent to the circumcircle of triangle DMP because of the tangent-secant theorem. Thus, from Power of a point in A we get that

$$AP^2 \overset{(DMP)}{=} AD \cdot AM = AI \cdot AI_A$$

which implies that line AP is tangent to Γ. Similarly, we have that AQ is tangent to Γ. Hence A, P, O' and Q must lie on a circle, namely circle Ω, which was what we wanted to show.



Solution 2. Let Ω denote the circumcircle of triangle ABC and let σ denote its incircle.

By symmetry, the segment PQ is tangent to σ. Moreover, by Poncelet's Porism, there exists a point R on Ω such that σ is the incircle of triangle PQR.

Thus, I is the incenter of triangle PQR, so the circumcenter of triangle PIQ must lie on the circumcircle of triangle PQR, which is just Ω.

Solution 3. Let $\Omega = \Omega_1$, Ω_2 and Ω_3 denote the circumcircles of triangles ABC, BIC and PIQ, respectively, and let O_1, O_2 and O_3 denote their respective circumcenters.

Notice that O_2 lies on Ω_1.

By the Radical Axis Theorem on Ω_1, Ω_2 and Ω_3, we have that the line IO_2 must be the radical axis of circles Ω_2 and Ω_3. Thus, line O_2O_3 is perpendicular to line IO_2, which implies that $\angle AO_2O_3 = \frac{\pi}{2}$.

Lastly, since line AO_1 is the reflection of the height from A to BC over line AI, we get that A lies on the perpendicular bisector of segment PQ. Thus, points A, O_1 and O_3 must be collinear.

Hence, point O_3 must be the antipodal point of A on Ω_1. I.e. the circumcenter of triangle PIQ lies on Ω, as desired.

Problem 12. Determine all real numbers θ for which there exists an infinite sequence $(x_n)_{n=1}^{\infty}$ of positive reals satisfying

$$x_{n-1} = x_n \cdot n^{x_n} \quad \text{and} \quad \frac{x_1}{n^\theta} \geq x_n$$

for all positive integers $n > 1$.

Solution. We claim that there exists an infinite sequence $(x_n)_{n=1}^{\infty}$ of positive reals which satisfies the given conditions if and only if $\theta \leq 1$.

Let $(\star)$ denote the equation $x_{n-1} = x_n \cdot n^{x_n}$ and let $(\dagger)$ denote the inequality $\frac{x_1}{n^\theta} \geq x_n$.

Lemma 1. If $\theta > 1$ there exists no sequences $(x_n)_{n=1}^{\infty}$ of positive reals satisfying $(\star)$ and $(\dagger)$.

Proof of Lemma 1. Suppose that there existed some real $\theta > 1$ and an infinite sequence $(x_n)_{n=1}^{\infty}$ of positive reals satisfying $(\star)$ and $(\dagger)$. Thus, for all positive integers $n > 1$ we have that

$$n^\theta \overset{(\dagger)}{\leq} \frac{x_1}{x_n} = \prod_{k=2}^{n} \frac{x_{k-1}}{x_k} \overset{(\star)}{=} \prod_{k=2}^{n} k^{x_k} \overset{(\dagger)}{\leq} \prod_{k=2}^{n} k^{x_1 k^{-\theta}}.$$

Taking the logarithm of both sides yields

$$\theta \log(n) \leq x_1 \sum_{k=2}^{n} \frac{\log(k)}{k^\theta}.$$

Notice that

$$\lim_{n\to\infty} x_1 \sum_{k=2}^{n} \frac{\log(k)}{k^\theta} < \infty$$

since $\frac{\log(k)}{k^\theta} \leq k^{-\frac{\theta+1}{2}}$ for all sufficiently large k (and $-\frac{\theta+1}{2} < -1$).

However, we also have that

$$\lim_{n\to\infty} \theta \log(n) = \infty$$

which yields a contradiction.

Thus, there are no sequences $(x_n)_{n=1}^{\infty}$ of positive reals satisfying $(\star)$ and $(\dagger)$ for $\theta > 1$. □

Lemma 2. For all $\theta \leq 1$ there exists sequences $(x_n)_{n=1}^{\infty}$ of positive reals satisfying $(\star)$ and $(\dagger)$.

Proof of Lemma 2. We claim that there exists a sequence $(x_n)_{n=1}\infty$ of positive reals such that

$$x_{n-1} = x_n \cdot n^{x_n} \quad \text{och} \quad x_1 \geq n x_n$$

for all positive integers $n > 1$.

Notice that the equation $y = x \cdot n^x$ always has a positive solution for x for any $y \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Therefore there exists a sequence (x_n) satisfying $(\star)$ for which

$$x_1 \geq \max_{n\geq 2} \left\{ n \log_n \left(\frac{n}{n-1} \right) \right\}.$$

[*The right-hand side above is finite since*

$$n \log_n \left(\frac{n}{n-1} \right) = \frac{\log\left(\left(1 + \frac{1}{n-1}\right)^n \right)}{\log(n)} \leq \log\left(\left(1 + \frac{1}{n-1}\right)^{2(n-1)} \right) < 3$$

for all sufficiently large n.]

Suppose that this sequence does not satisfy that $x_1 \geq n x_n$ for all $n \in \mathbb{N}$. Then there exists a minimal $N \in \mathbb{N}$ for which $x_1 < N x_N$. (Notice that we necessarily have $N \geq 2$). Thus we have that

$$\frac{x_1}{N-1} \geq x_{N-1} \overset{(\star)}{=} x_N \cdot N^{x_N} > \frac{x_1}{N} \cdot N^{\frac{x_1}{N}}$$

$$\therefore N \log_N \left(\frac{N}{N-1} \right) > x_1$$

which contradicts our choice of x_1.

Hence, the constructed sequence $(x_n)_{n=1}^{\infty}$ satisfies both $(\star)$ and $x_1 \geq n x_n$ (and therefore also $(\dagger)$, since $\theta \leq 1$) for all positive integers $n > 1$, which was what we wanted to show. □

We conclude that there exists infinite sequences $(x_n)_{n=1}^{\infty}$ of positive reals satisfying $(\star)$ and $(\dagger)$ for all positive integers $n > 1$ if and only if $\theta \leq 1$.

Problem 13. Emil has n stones whose weights are $1, 2, \dots, n$ kilograms. Yesterday he attached a label to each stone showing its weight, but he is worried that his friend Ivar played a prank and swapped the labels during the night. Emil wants to determine whether the labels are correct or not by performing a number of weighings on his balance scale. After each weighing he is told which pan is heavier, or that both weigh the same. Can you come up with some weighings Emil can perform that are guaranteed to expose Ivar if he moved the labels?
You receive more points the fewer weighings your solution uses for large n!

Solution. As the problem description suggests, this is a fairly open-ended problem with multiple possible solutions. Here we present a very efficient method that requires only

$$2\lceil \log_2(n) \rceil + 1$$

weighings. We say that a stone is "determined" if Emil knows that the label on it is correct.

Start by weighing $1 + 2$ against 3, then $1 + 2 + 3$ against 6, then $1 + 2 + 3 + 6$ against 12, and so on. At each step, weigh all stones used so far against the stone that should weigh as much as their total. This way, the number on the stone on the right pan doubles in each step until the sum of all previously weighed stones exceeds n. The number on the last stone weighed will be $3 \cdot 2^{\lfloor \log_2(n/3) \rfloor}$. All labels on stones weighed so far — let's call this set X — can now be combined to form any integer weight between 1 and $6 \cdot 2^{\lfloor \log_2(n/3) \rfloor}$ (think of binary numbers, multiplied by three). In particular, if we haven't weighed stone n, we can pick stones from X whose labeled weights sum to n and weigh them against the stone labeled n, and we do just that. In total, we have now used at most

$$\lfloor \log(\frac{n}{3}) \rfloor + 2$$

weighings. If any of the weighings result in the two pans being unequal, Emil immediately knows that some labels have been swapped. The same logic holds throughout, so from now on we assume every weighing gives the expected result. If stones 1 and 2 are correctly labeled, then every other stone weighed so far (i.e., all of X and the one labeled n) must also be correct. If 1 and 2 are mislabeled, they must either have been swapped with each other or their combined weight is more than 3, which would make the total weight of all weighed stones exceed the labeled values — which is impossible. However, we still need to ensure that stones 1 and 2 aren't just swapped, which can easily be verified in a single weighing. So now we have determined the correct labels of all stones in X and the stone labeled n.

Let Y be the set of stones that are still undetermined. We now show how to determine the identity of each of these in $\lceil \log_2(n) \rceil$ steps. We use the fact that an integer $\leq n$ is uniquely determined by its first $\lceil \log_2(n) \rceil$ binary digits. For each integer $k \leq \lceil \log_2(n) \rceil$, let L_k and U_k denote the set of stones in Y whose k-th binary digit is 0 and 1, respectively. The rest of the algorithm is best described in pseudocode:

for $k = 1, \dots, \lceil \log_2 n \rceil$: **do**
  if $\text{sum}(L_k) < \text{sum}(U_k)$: **then**
    Place all stones in L_k beside pan 1 (Don't put them on there just yet)
    while total weight beside pan 1 > pan 2: **do**
      Put a stone from U_k beside pan 2, starting with the one with the **highest** label.
    end while
    Let D be how much more the stones besides pan 2 will weigh than those beside pan 1 if all labels are correct. Note: $D < n$.
    Pick stones from X with total weight D and put them beside pan 1.
    Perform the weighing!
  else
    Place all stones in U_k beside pan 1 (Don't put them on the pan just yet)
    while total weight beside pan 2 > pan 1: **do**
      Put a stone from L_k beside pan 1, starting with the one with the **highest** label.
    end while
    Let D be how much more the stones besides pan 1 will weigh than those beside pan 2 if all labels are correct. Note: $D < n$.
    Pick stones from X with total weight D and put them beside pan 2.
    Perform the weighing!
  end if
end for

Claim: After k iterations of the algorithm, Emil has determined the first k binary digits of every stone's weight.

Proof of claim: By induction. Assume that after $k-1$ iterations, Emil knows the first $k-1$ binary digits of each stone's weight. Then in iteration k, pan 1 is as light as possible if the stones on it are correctly labeled, and pan 2 is as heavy as possible if the stones on it are correctly labled. Thus, if the set of stones in each pan is not as the lables suggest, then pan 1 would weigh more than pan 2.

If both pans weigh equally — which should happen if the labels are correct — then Emil knows which stones are in each pan. In Case 1, this means he knows exactly which are in L_k; in Case 2, he knows exactly which are in U_k. In either case, Emil now knows which stones in Y have a 0 and which have a 1 in their k-th binary digit, proving the claim. □

After k iterations, Emil knows the first k binary digits of each weight, and thus can verify whether the labels are correct. The total number of weighings used is:

$$\begin{aligned}
&\lceil \log_2(\frac{n}{3}) \rceil + 2 + \lceil \log_2(n) \rceil \\
\leq &\lceil 2 \log_2(n) \rceil + 1.
\end{aligned}$$

This is not an optimal solution but it is impossible to solve the problem using fewer than $\lceil \log_3(n) \rceil$ weighings. The reason is that each weighing can divide the stones into at most three groups: those on pan 1, those on pan 2, and those not weighed. To ensure that no two labels are swapped, every pair of stones must appear in different groups in at least one weighing, which requires at least $\lceil \log_3(n) \rceil$ weighings.