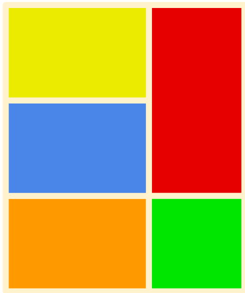
HÖJD PUNKTEN

UVS MATEMATIKER

Open competition 15th to 17th of May 2025

Time allowed: 5 hours
Allowed aids: pens, eraser, compas and ruler only
Each problem is worth 7 points.
Full credit requires justification unless otherwise stated.

Problem 1. A rectangle is partitioned into n smaller rectangles which are to be painted in different colors. In preparation, strips of tape need to be placed covering all of the edges of the rectangles such that no two strips of tape cross. What is the fewest number of strips of tape needed?



Problem 2. Let n and k be positive integers with $n \geq \max\{k, 3\}$. Alice and Bob play a game on an (undirected and simple) graph G. At the start of the game, G is a complete graph on n nodes. Each round, Alice first selects k edges in G, and then Bob removes one of the edges Alice had selected.

The game ends when G only has $n-1$ edges left. Then Alice wins if G is connected and otherwise Bob wins. What is the largest positive integer k, in terms of n, for which Alice wins if both players play optimally?

Problem 3. A frog is positioned at the point $(0, 0)$ in the plane and starts jumping. The frog first performs a jump of length 1, and all its subsequent jumps are double the length of its previous jump. All jumps are parallel to the coordinate axes. Which points of the plane can the frog reach by jumping in this way?

Problem 4. A stick of length 1 is split between a countable number of people $P_1, P_2, \ldots$. First, P_1 chooses a number U_1 uniformly from $[0, 1]$ and the stick is split in a piece of length U_1 and a piece of length $1 - U_1$. P_1 keeps the stick of length U_1 and passes to P_2 the remaining piece. P_2 then chooses U_2 uniformly from $[0, 1]$, and splits the stick into two pieces of length $U_2(1 - U_1)$ and $(1 - U_2)(1 - U_1)$. The first piece is kept by P_2 and the other stick is given to P_3, and the process continues so that P_n recieves a stick of length $U_n(1 - U_{n-1}) \ldots (1 - U_1)$. What is the probability that the stick that every person recieves is shorter than $1/2$?

Problem 5. Lisa the astronaut is constructing humanity's first vegetable plantation on planet Mars. She is building it out of a grid of hexagonal plantation modules whose side lengths are all 1 meter long. Right above the middle of the middle plantation module, she places a light source that illuminates all modules within a circle of radius 100m. For a plantation module to work properly, it must be completely illuminated. Given that Lisa has filled the illuminated circle with as many plantation modules as possible, determine the perimeter of the plantation.

Problem 6. Let ABC be a scalene triangle with incenter I and circumcircle Ω. Let M be the midpoint of arc BC on Ω containing A. Let D be the intersection of lines BC and AM. Let J be the second intersection of line DI with the circumcircle of triangle BIC. Prove that the tangent to the circumcircle of triangle BIC at J bisects segment DM.

Problem 7. Let n be a positive integer. Oscar receives a bag containing n distinct positive integers and writes on a board all possible numbers of the form $xy + z$, where x, y and z are (not necessarily distinct) numbers from the bag. Given n, determine the minimum number of *distinct* numbers that Oscar may have written on the board.

Problem 8. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying that

$$d\left(\sum_{i=1}^{N} a_i\right) \leq d\left(\sum_{i=1}^{N} f(a_i)\right)$$

for all N, $a_1, \ldots, a_N \in \mathbb{N}$. (Here, $d(n)$ denotes the number of divisors of n.)

Problem 9. Let $a_1, a_2, a_3, \ldots,$ be an infinite sequence of distinct positive integers, and let N be a positive integer. Suppose that, for each integer $n > N$, a_n is equal to the smallest positive integer which cannot be written as a sum of distinct elements of $\{a_1, \ldots, a_{n-1}\}$.

Prove that there exists a positive integer M such that $a_m = 2a_{m-1}$ for all $m > M$.

Problem 10. Let P be a non-constant polynomial with integer coefficients and positive leading coefficient. For each positive integer n, prove that there exists a positive integer c such that $P(x)+c$ is prime for at least n different integers x.

Problem 11. Let ABC be a triangle with incenter I and circumcircle Ω. Let the reflection of line BC over line AI intersect Ω at points P and Q. Prove that the circumcenter of triangle PIQ lies on Ω.

Problem 12. Determine all real numbers θ for which there exists an infinite sequence $(x_n)_{n=1}^{\infty}$ of positive reals satisfying

$$x_{n-1} = x_n \cdot n^{x_n} \quad \text{and} \quad \frac{x_1}{n^{\theta}} \geq x_n$$

for all positive integers $n > 1$.

Problem 13. Emil has n stones whose weights are $1, 2, \ldots, n$ kilograms. Yesterday he attached a label to each stone showing its weight, but he is worried that his friend Ivar played a prank and swapped the labels during the night. Emil wants to determine whether the labels are correct or not by performing a number of weighings on his balance scale. After each weighing he is told which pan is heavier, or that both weigh the same. Can you come up with some weighings Emil can perform that are guaranteed to expose Ivar if he moved the labels?
You receive more points the fewer weighings your solution uses for large n!