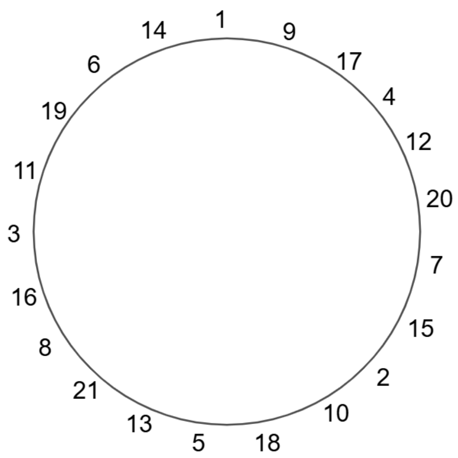
Middle School Competition on May 16th 2025

Solutions

Problem 1. Write the numbers $1, 2, \dots, 21$ in a circle in some order so that the difference between any two adjacent numbers is always either 8 or 13. *(Answer only required.)*

Solution. We can arrange the numbers as follows:



Problem 2. Over Friday, Saturday, and Sunday John ate some grapes. He ate the same number of red grapes each day, and in total over the three days he ate as many red grapes as green grapes. If John ate 5 grapes on Friday and 18 grapes on Saturday, how many *green* grapes did he eat on Sunday?

Solution. The important question one should ask oneself for this question is: How many red grapes did John eat each day? We let x be this number. The following table shows how many red and green grapes that John ate each day:

	Friday	Saturday	Sunday
Red	x	x	x
Green	$5 - x$	$18 - x$	?

We see that x can be at most 5, since John ate $5 - x$ green grapes on Friday. Moreover, if x is at most 4 then the total number of red grapes John ate ($3x$) is less than the number of green grapes he ate on Saturday ($18 - x$). This is impossible, since we are given that John ate an equal number of red and green grapes in total. Therefore the only remaining case is that John ate $x = 5$ red grapes each day. Then, he must have ate 0 green grapes on Friday, 13 green grapes on Saturday, and the remaining 2 green grapes on Sunday.

Problem 3. This year Höjdpunkten has a logo made of twelve squares that together form a rectangle (see the top of the page). If the height of this rectangle is 11 units, how wide is it?

Solution. The squares surrounding the letters H, Ö, J and D have the same side length, which we will call x. Similarly, the squares surrounding the letters P, U, N, K, T, E and N have the same side length, which we will call y. For the squares to align correctly vertically, we must have

$$x + y = 11 \implies y = 11 - x.$$

For them to align horizontally, we must have

$$\begin{aligned} & 4x = 7y \\ \implies & 4x = 7 \cdot (11 - x) = 77 - 7x \\ \implies & x = \frac{77}{4+7} = \frac{77}{11} = 7 \end{aligned}$$

Therefore, the logo must be $11 + 4 \cdot 7 = 39$ units wide.

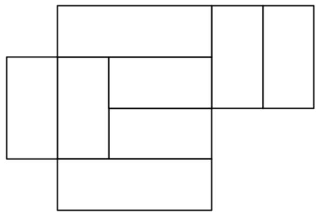
Problem 4. Sofia has a balance scale and nine weights of $1, 2, 4, 8, 16, 32, 64, 128$, and 141 grams. How can she balance all the weights on the scale with none left over? *(Answer only required.)*

Solution.

Bowl 1	141 g, 32 g, 16 g, 8 g, 1 g
Bowl	128 g, 64 g, 4 g, 2 g

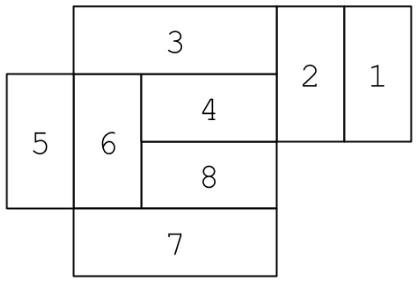
One can, for example, arrive at this construction by placing the weights on the scale in order from heaviest to lightest and always placing the next weight in the bowl which weighs the least at each step.

Problem 5. We call two numbers *difference-divisible* if both are divisible by their difference. Place the numbers 1 through 8 in the figure below so that numbers in adjacent rectangles are difference-divisible. *(Answer only required.)*

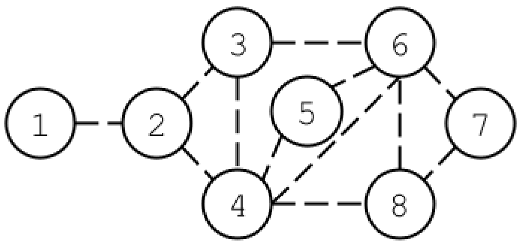


For example, 10 *and* 20 *are difference-divisible because* $20 - 10 = 10$ *divides both numbers. By contrast,* 17 *and* 25 *are not difference-divisible because neither is divisible by* $25 - 17 = 8$.

Solution. The numbers can be arranged as follows:



Comment: It is possible to prove that this is the only possible solution. We can do this by drawing a graph (a network) describing which pairs of numbers can be placed next to each other.



We start by noting that the number 1 only has one neighbor, and must therefore be placed in the rectangle furthest to the right or to the left. If it's placed in the rectangle furthest to the left then the number 2 must be placed in the rectangle to the right of it, which is not possible since the number 2 has at most three neighbors, but the rectangle has five neighbors. Therefore, the numbers 1 and 2 must be placed as they are in the solution above.

Next to the number 2 we must place the numbers 3 and 4. Since the number 3 has at most three neighbors it must be placed in the wide rectangle on top. Therefore the numbers 3 and 4 must be placed as they are in the solution above.

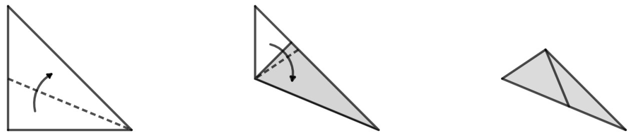
The third neighbor to the number 3 can only be the number 6. Therefore the number 6 must be placed as it is in the solution above.

Consider the rectangle which neighbors both numbers 4 and 6. The only numbers which can have both 4 and 6 as neighbors are 3 and 8, but notice that the number 3 has already has its place fixed in the figure. Therefore the number 8 must be placed as it is in the solution above.

Consider the bottom-most rectangle which neighbors both numbers 6 and 8. The only numbers which can have both 6 and 8 as neighbors are 4 and 7, but notice that the number 4 has already has its place fixed in the figure. Therefore the number 7 must be placed as it is in the solution above.

Left over is the number 5, which is forced to be placed in the left-most rectangle.

Problem 6. Theodor cuts a square sheet of paper in half along a diagonal to make a triangle. He then folds the paper twice as shown in the picture so that it becomes a smaller triangle. Determine all three angles of the new triangle.



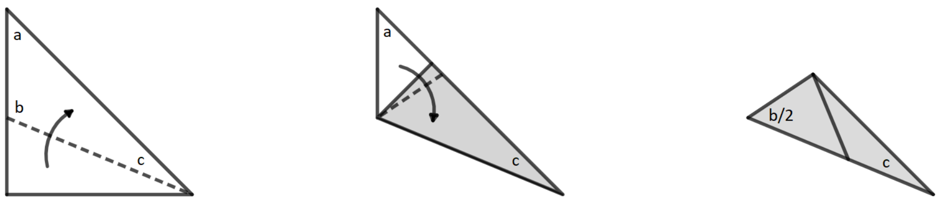
Solution. The first triangle's angles are 90°, 45° and 45°. After one fold, the angle at one of the 45°-corners is halved to create the angle $c = \frac{45^\circ}{2} = 22.5^\circ$. Since the sum of the angles of a triangle equals 180°, we can compute b as:

$$b = 180^\circ - a - c = 180^\circ - 45^\circ - 22.5^\circ = 112.5^\circ.$$

Afterwards, the second fold halves the angle b, which results in the angle $\frac{b}{2} = \frac{112.5^\circ}{2} = 56.25^\circ$. The last angle is

$$180^\circ - \frac{b}{2} - c = 180^\circ - 56.25^\circ - 22.5^\circ = 101.25^\circ.$$

Answer: 22.5°, 56.25° and 101.25°.



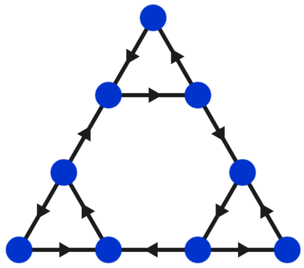
Problem 7. Emil has six stones whose weights are $1, 2, 3, 4, 5$, and 6 kilograms. Yesterday he put a label showing its weight on each stone, but he worries that his friend Ivar played a prank overnight and swapped the labels. Emil wants to decide whether the labels are correct by performing a number of weighings on his balance scale. After each weighing he learns which pan is heavier, or that they have the same weight. Can you suggest weighings Emil can perform that are guaranteed to expose Ivar if he moved the labels?
The fewer weighings your solution uses in the worst case, the more points you receive!

Solution. There exists multiple solutions which use three weighings. However, there is one way to solve the problem using only two weighings:

1. Place stones 1, 2 and 3 in the left bowl and stone 6 in the right bowl. If the bowls don't weigh the same, then the lables must have been moved around. If they weight the same, then the stones in the left bowl must weigh 1, 2 and 3 kilograms in some order and the stone in the right bowl must weigh 6 kilograms, because otherwise the left bowl would have been heavier than the right one.

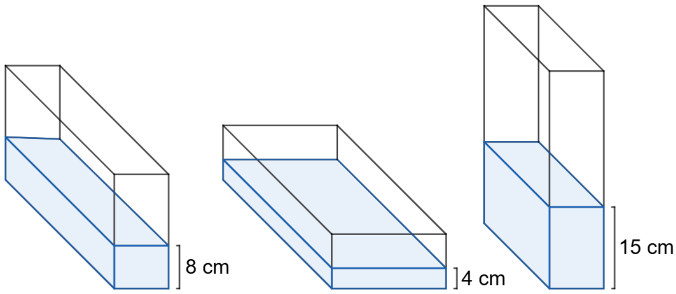
2. It remains to determine which of the stones 1, 2 and 3 weight 1, 2 and 3 kilograms as well as which of the stones 4 and 5 weight 4 and 5 kilograms. This can be done by placing stone 3 and 5 in the left bowl and stone 1 and 6 in the right bowl. If the left bowl weighs more than the right bowl, then the lables must have been correctly placed. This must be the case since stone 3 and 5 can weigh at most 8 kilograms at most and stone 1 and 6 must weight at least 7 kilograms.

Problem 8. In a trivia game the board looks as in the figure below. You start on some square. Each round you move one step along an arrow, in the direction the arrow points. Is it possible to return to the square you started on after exactly 20 moves?



Solution. Consider some path which begins and ends at the same square. The number of steps in the path increases by 6 every time you go around the middle hexagon and by 3 every time you go around one of the smaller triangles. But it is impossible to get 20 from adding 6:es and 3:s together, since 20 is not divisible by 3. Therefore, it is impossible to return to the square you started on after exactly 20 moves.

Problem 9. Cecilia pours three litres of water into a rectangular glass container with a lid. When she places it on a table, the water depth can be 8 cm, 4 cm, or 15 cm depending on how she rotates it. What is the volume of the container?



Solution. Let x, y and z denote the container's side lengths, where $x < y < z$. The water's volume (3000 milliliters) can then be calculated in three different ways, which gives

$$3000 = 8xz = 4yz = 15xy.$$

That $8xz = 4yz$ means that $y = 2x$. Therefore we have

$$\begin{aligned} 3000 &= 15xy = 30x^2 \\ \implies x^2 &= 3000/30 = 100 \\ \implies x &= 10 \\ \implies y &= 2x = 20 \\ \implies z &= \frac{3000}{8 \cdot 10} = 37.5 \end{aligned}$$

The container's volume thus becomes

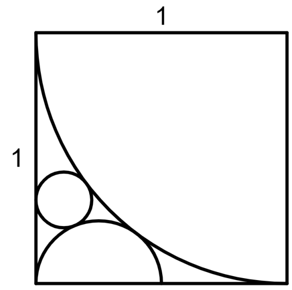
$$xyz = 10 \cdot 20 \cdot 37.5 = 7500 \text{ ml}.$$

Problem 10. A *palindromic number* reads the same forwards and backwards, for example 494. If you multiply together all three-digit palindromic numbers, how many zeros does the product end with?

Solution. The number of trailing zeroes at the end of a number is determined by how many times that number is divisible by $10 = 5 \cdot 2$. Amongst all three-digit palindromic numbers there are very many which are divisible by 2 (i.e. are even), so it suffices to count the number of times the product of all three-digit palindromic numbers is divisible by 5. The only three-digit palindromic numbers which are divisible by 5 are: 515, 525, 535, 545, 555, 565, 575, 585, 595 and 505 (10 numbers in total). Amonst these only 525 and 575 are divisible by 5 multiple times, in particular twice, whilst the remaining 8 are only divisible by 5 once. Therefore, the product of all three-digit palindromic numbers is divisible by 5 exactly $8 + 2 \cdot 2 = 12$ times.

Answer: 12 zeroes.

Problem 11. A quarter-circle, a semicircle, and a full circle are packed into a square of side length 1 unit, as in the figure below. Determine the diameter of a) the semicircle b) the full circle.



Solution. (a) Let r be the half-circle's radius. From Pythagoras Theorem we get that

$$\begin{aligned}&(1-r)^2+1^2=(1+r)^2\\ &\implies 1-2r+r^2+1=1+2r+r^2\\ &\implies 1=4r\\ &\implies r=\frac{1}{4}.\end{aligned}$$

(b) Let (x, y) be the position of the center of the small circle, where we take the origin to be the bottom left corner of the square. Since the circle is tangent to the square's left edge, the circle's radius will be exactly x. The distance from the circle's center to the the centers of the quarter- and half-circles must be $x+1$ and $x+r$, respectively, so by Pythagoras Theorem we get that

$$\begin{cases}(r-x)^2+y^2=(r+x)^2\\ (1-x)^2+(1-y)^2=(1+x)^2\end{cases}$$

If we expand, simplify and use that $r=\frac{1}{4}$, we get that

$$\begin{cases}r^2-2rx+x^2+y^2=r^2+2rx+x^2\\ 1-2x+x^2+1-2y+y^2=1^2+2x+x^2\end{cases}$$

$$\iff \begin{cases}y^2=4rx=x\\ (1-y)^2=4x\end{cases}$$

Therefore $(1-y)^2=4y^2$, since both $(1-y)^2$ and $4y^2$ are equal to $4x$. We can now take the square root of both sides to obtain

$$1-y=2y \implies y=\frac{1}{3}.$$

Finally, we get that the radius of the small circle is $x=y^2=\frac{1}{9}$.

Problem 12. A frog is at the point $(0,0)$ in the plane and starts jumping. Its first jump has length 1, and each subsequent jump is twice as long as the previous one. Every jump is made parallel to one of the coordinate axes. Which points can the frog reach by jumping in this way?

Answer: All points (x,y) where $x+y$ is odd (and also the origin).
Note: Another solution can be found in the solution document for the high-school competition

Solution. Let us call a point (x,y) odd/even if $x+y$ is odd/even. After the first jump, the frog will be at one of the points $(0,1)$, $(0,-1)$, $(1,0)$ or $(-1,0)$. All these points are odd. All other jumps which the frog takes will change $x+y$ by an even number, so therefore the frog will never reach an even point except for $(0,0)$ where it started.

Let K_n be the square with vertices at the four points $(-2^n,0),(2^n,0),(0,-2^n),(0,2^n)$. We claim that the points that the frog can reach after exactly n jumps are precisely the odd points that lie inside K_n. For $n=1$ claim holds (draw a picture). Suppose that the claim holds for $n=p$. We now wish to show that it also holds for $n=p+1$. Since the statement holds for $n=p$ we know that the frog can reach all odd points inside K_p using p jumps. Let Låt $K_p^{\uparrow}, K_p^{\downarrow}, K_p^{\leftarrow}, K_p^{\rightarrow}$ be the four squares that are obtained by shifting K_p up, down, left, and right 2^p steps. The points the frog can reach after $p+1$ jumps are precisely the odd points that lie in one of $K_p^{\uparrow}, K_p^{\downarrow}, K_p^{\leftarrow}$, or $K_p^{\rightarrow}$ depending on in which direction the last jump was made. Together these four smaller squares form the larger square K_{p+1} which proves that the claim holds for $n=p+1$. By induction, we now know that the claim holds for all n and thus the frog can reach all odd points.