HÖJD PUNKTEN

UVS MATEMATIKER

High School Competition on May 15th 2025

Solutions

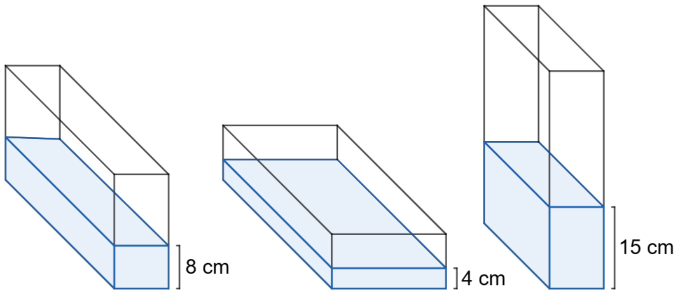
Problem 1. Sofia has a balance scale and nine weights weighing $10, 20, 40, 80, 160, 320, 640, 1280$, and 1410 grams, respectively. How can she balance these weights on the scale without any left over? (*Only an answer is required.*)

Solution.

Pan 1	1410 g, 320 g, 160 g, 80 g, 10 g
Pan 2	1280 g, 640 g, 40 g, 20 g

One can come up with this solution for example by placing the weights on the scale form largest to smallest, always placing the next weight on the lightest side.

Problem 2. Cecilia pours three litres of water into a rectangular glass container with a lid. When she places it on a table, the water level becomes 8 cm, 4 cm or 15 cm depending on how she rotates it. What is the volume of the container?



Solution. Let the rectangular glass container have dimensions a cm $\times$ b cm $\times$ c cm.

Since the volume of the water in the container is 3 L $=$ 3000 cm^3, we have that

$$\begin{cases} 8ab = 3000 \\ 4bc = 3000 \\ 15ca = 3000 \end{cases}.$$

Multiplying all three of these equations together gives

$$\begin{aligned} 8ab \cdot 4bc \cdot 15ca &= 3000^3 \\ \Longrightarrow 2^5 \cdot 3 \cdot 5 \cdot (abc)^2 &= 2^9 \cdot 3^3 \cdot 5^9 \\ \Longrightarrow (abc)^2 &= 2^4 \cdot 3^2 \cdot 5^8 \\ \Longrightarrow abc &= 2^2 \cdot 3 \cdot 5^4 = 7500 \end{aligned}$$

so the volume of the container is a cm $\times$ b cm $\times$ c cm $=$ 7500 cm^3 $=$ 7.5 L.

Problem 3. A palindromic number is a number that reads the same backwards as forwards, for example 494. If you multiply together all three–digit palindromic numbers, how many zeros does the product end with?

Solution. The number of zeros at the end of the product is equal to the number of times 10 divides the product. Note that $10 = 5 \cdot 2$, so we need to find how many times 5 and 2 divide the product.

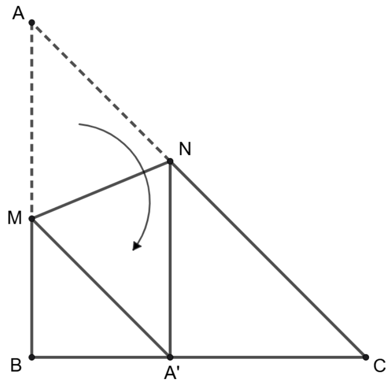
Note that the palindromes of the form $5X5$, where $0 \leq X \leq 9$, are the only three-digit palindromic numbers divisible by 5. For $X \neq 2, 7$, these palindromes are divisible by 5 exactly once, and for $X = 2, 7$, they are divisible by 5 exactly twice. Therefore, the product of all three-digit palindromic numbers is divisible by 5 twelve times.

Note that the product of all three-digit palindromic numbers is divisible by 2 at least twelve times, since all palindromes of the forms $2X2$ and $4X4$, where $0 \leq X \leq 9$, are each divisible by 2 at least once.

Therefore, the product of all three-digit palindromic numbers ends with exactly

Answer twelve zeros.

Problem 4. Theodor makes a triangle by halving a $1\,\mathrm{dm} \times 1\,\mathrm{dm}$ square sheet of paper along the diagonal. He then folds corner A to the point A' so that the crease MN is formed (where M and N lie on AB and AC, respectively) and so that $A'N$ is perpendicular to BC. In what ratio does A' divide the segment BC?



Solution. The length of AN does not change during the folding, so we have

$$AN = A'N = \sin(45^\circ) \cdot NC = \sqrt{2} \cdot CN \iff \frac{CN}{AN} = \sqrt{2}.$$

By the intercept theorem (transversal theorem), it follows that

$$\frac{CA'}{BA'} = \frac{CN}{AN} = \sqrt{2}.$$

Answer: A' divides the segment BC into parts BA' and CA' in the ratio $1 : \sqrt{2}$.

Problem 5. A frog is at the point $(0, 0)$ in the plane and starts jumping. Its first jump has length 1, and each subsequent jump is twice as long as the previous one. Every jump is made parallel to one of the coordinate axes. Which points can the frog reach by jumping in this way?

Solution. Notice that the frog is only able to reach points whose integer coordinates (x, y) satisfy that $x + y$ is odd. We claim that the frog can reach all of these points.

Lemma 1. Let $n \in \mathbb{N}$. The frog can jump to all points (x, y) with integer coordinates such that $x + y$ is odd and $|x| + |y| \leq 2^n - 1$ in exactly n jumps.

Proof of Lemma 1. We will prove the lemma by induction on n.

Base case ($n = 1$): On its first jump, the frog can jump to $(1, 0)$, $(0, 1)$, $(-1, 0)$ or $(0, -1)$, which encompasses all points (x, y) with integer coordinates such that $x + y$ is odd and $|x| + |y| \leq 2^1 - 1$.

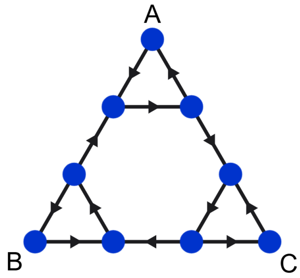
Induction step ($n > 1$): Notice that the frog can jump to all points (x', y') with integer coordinates such that $x' + y'$ is odd and $|x'| + |y'| \leq 2^{n-1} - 1$ using exactly $n - 1$ jumps. Consider a point (x, y) with integer coefficients such that $x + y$ is odd and $|x| + |y| \leq 2^n - 1$. We have $x \neq y$ since $x + y$ is odd. By symmetry, we can assume that $x > y \geq 0$. The point $(x - 2^{n-1}, y)$ is then odd since $n > 1$ and we have

$$\begin{cases} x \geq 2^{n-1} \Rightarrow |x - 2^{n-1}| + |y| = x - 2^{n-1} + y \leq 2^n - 1 - 2^{n-1} = 2^{n-1} - 1 \\ x < 2^{n-1} \Rightarrow |x - 2^{n-1}| + |y| = 2^{n-1} - x + y = 2^{n-1} - (x - y) \leq 2^{n-1} - 1. \end{cases}$$

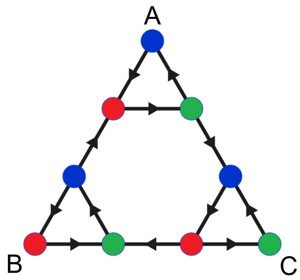
By the induction hypothesis, the frog can reach $(x - 2^{n-1}, y)$ after exactly $n - 1$ moves and can thus reach (x, y) after exatcly n jumps (the n:th jump is of length 2^{n-1}). Thus, we have shown that the frog can jump to all points (x, y) with integer coordinates such that $x + y$ is odd and $|x| + |y| \leq 2^n - 1$ using exactly n jumps, which implies the induction step. □

Therefore the set of points to which the frog can jump to is exactly the set of points (x, y) with integer coordinates such that $x + y$ is odd.

Problem 6. In a trivia game, the board looks like in the figure below. Three players, A, B, and C start in three different corners of the board. In each round they then take one step along an arrow. They are only allowed to move in the direction the arrow points. In the end all players have taken 20 steps. Prove that all players finish on different squares.



Solution. The squares can be coloured red, blue, and green as in the figure below:



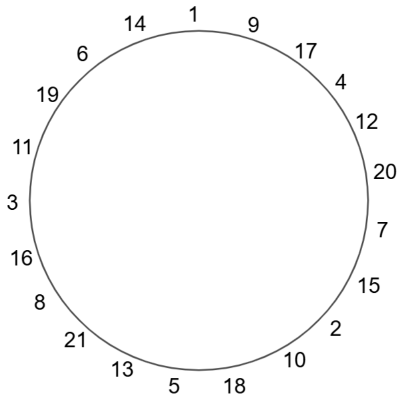
Each move, one must go from blue to red, from red to green, or from green to blue. All players start at different coloured squares and will thus continue to be at different coloured squares after every player has made a move. After each palyer has made 20 moves, the players will still be at different coloured squares, and thus they are all at different squares after 20 moves.

Problem 7. The Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ are the sequence obtained by starting with two ones and then computing each subsequent term by adding the two previous terms.

(a) Write the integers $1, 2, 3, \ldots, 20, 21$ in a circle in some order such that the difference between two adjacent numbers is always either 8 or 13.

(b) Let a, b and n be three consecutive Fibonacci numbers. Prove that it is possible to write the numbers $1, 2, \ldots, n$ in a circle so that the difference between two adjacent numbers is always either a or b.

Solution. (a)



(b) (Solution 1) We use induction.
Induction hypothesis Suppose that the for consecutive Fibonacci numbers a, b, n, the numbers $1, \ldots, n$ can be placed in a circle such that the difference between any two adjacent numbers is either a or b. **Base case:** For $a = 1, b = 1, n = 2$, the statement is clearly true.
Induction step The next Fibonacci number after n is $n + b$. We write the numbers $1, \ldots, n$ along a circle in accordance with the induction hypothesis. We now notice that

- For all $k = 1, \ldots, a$, $k - b < 1$ and $k - a < 1$. Thus, k must have $k + a$ and $k + b$ as its neighbours.
- For all $k = a + 1, \ldots, b$, $k - b < 1$ and $k + b > n$. Thus k must have $k - a$ and $k + a$ as its neighboors.
- For all $k = 1, \ldots, n$, $k + a > n$ and $k + b > n$. Thus k must have $k - a$ and $k - b$ as its neighbours.

All numbers $k = 1, \ldots, b$ must therefore have $k + a$ as a neighbour. If we for all such k write the number $k + n$ between k and $k + a$ we will now have every number between 1 and $b + n$ on the circle exactly once. Furthermore, every difference between adjacent numbers is now b or n. This new circle satisfies the induction hypothesis for the next three consecutive Fibonacci numbers: $b, n, b + n$. Thus the induction step is done.

(b) (Solution 2) It is well known that consecutive Fibonacci numbers are coprime. Let a, b, n be consecutive Fibonacci numbers. Write the numbers in the order $s_1, \ldots, s_n$ where $s_i = 1 + (ib \,\%\, n)$ and $\%n$ denotes the remainder after division by n. Since b and n are coprime, the s_i are a permutation of $1, \ldots, n$. Looking at the differences $\pmod n$ we find

$$s_{k+1} - s_k \equiv (k + 1)b - kb \equiv b \pmod n.$$

Since all differences have absolute value less than n, the only two possible differences between adjacent numbers s_{k+1}, s_k are b and $-a$, which is what we wanted to prove.

Problem 8. The number $n = 10^9 + 7$ is a prime. The numbers $1, 2, 3, \ldots, n$ are placed on a circle in that order (so 1 is adjacent to n). A straight line is drawn that separates the numbers into two groups. Both groups have the same sum. Show that there is only one possible such division and find it.

Solution. For two adjacent numbers on the circle, their difference is 1 except for 1 and n. There will therefore be one side of the line which contatains a set of consecutive numbers. if we let a and b denote the smallest and the largest number on that side of the line, the sum of the numbers on the same side as a and b is

$$a + a + 1 + a + 2 + \cdots + b - 1 + b = \frac{1}{2}(a + b)(b - a + 1) \qquad (1)$$

(the sum of an arithmetic progression is the mean value times the number of terms) If both sides of the line should have the same sum, (1) must be half the sum of all numbers on the circle. This means that

$$\begin{aligned} \frac{1}{2}(a + b)(b - a + 1) &= \frac{1}{2}(1 + 2 + \cdots + n) \\ \iff 2(a + b)(b - a + 1) &= n(n + 1) \qquad (2) \end{aligned}$$

where we once again used the formula for the sum of an arithmetic progression. We see that n must divide $2(a+b)(b-a+1)$ and since n is a prime greater than 2, n must divide $a+b$ or $b-a+1$. For $b - a + 1$ to be divisible by n, we would need to have $a = 1, b = n$ which obviously does not work (all number would be on the same side of the line). Thus n must divide a+b. Since a and b are at most n, $a + b$ can only be equal to n or $2n$. The second case is impossible since this would imply that $a = b = n$, and then one side of the line would only contain the number n, and have way smaller sum than the side with all the other numbers. We can therefore conclude that $a + b = n$. Substituting this into 2 yields

$$\begin{aligned} 2(b - a + 1) &= n + 1 \\ \iff 2((n - a) - a + 1) &= n + 1 \\ \iff 4a &= 2(n + 1) - (n + 1) \\ \iff a &= \frac{n + 1}{4} \end{aligned}$$

which gives us

$$b = n - \frac{n + 1}{4} = \frac{3n - 1}{4}.$$

If we substitute these values of a and b into 1, we indeed get

$$\begin{aligned} \frac{1}{2}(a + b)(b - a + 1) &= \frac{1}{2}\left(\frac{3n - 1}{4} + \frac{n + 1}{4}\right)\left(\frac{3n - 1}{4} - \frac{n + 1}{4} + 1\right) \\ &= \frac{1}{2}\left(\frac{4n}{4}\right)\left(\frac{2n + 2}{4}\right) \\ &= \frac{n(n + 1)}{4} \end{aligned}$$

as desired.

Answer: The only possible way to draw a line that divides the numbers into two sets with equal sum is to have all numbers from $a = \frac{n+1}{4} = 2.5 \cdot 10^8 + 2$ up to $b = \frac{3n-1}{4} = 7.5 \cdot 10^9 + 5$ on one side of the line and the remaining numbers on the other side of the line.

Problem 9. Five points A, B, C, D and E lie on a circle in that order such that $|BA| = |BC|$, $|DA| = |DE|$ and $|BD| = |CE|$. Prove that $\angle BAD = 60^\circ$.

Solution. We may without loss of generality assume that the circle has radius 1, and that the points A, B, C, D and E lie in counter-clockwise order around the circle.

Let x be the length of the arc from A to B on the circle in counter-clockwise direction, y be the length of the arc from B to C on the circle in counter-clockwise direction, z be the length of the arc from C to D on the circle in counter-clockwise direction, w be the length of the arc from D to E on the circle in counter-clockwise direction and t be the length of the arc from E to A on the circle in counter-clockwise direction.

Notice that

$$x + y + z + w + t = 2\pi$$

and $0 < x, y, z, w, t < 2\pi$.

Since $|BA| = |BC|$, we have that

$$\left[\begin{array}{l} x = y \\ x = x + z + w + t \end{array}\right.$$
$$\therefore x = y.$$

Since $|DA| = |DE|$, we have that

$$\left[\begin{array}{l} x + y + z = w \\ x + y + z = t + x + y + z \end{array}\right.$$
$$\therefore x + y + z = w.$$

Since $|BD| = |CE|$, we have that

$$\left[\begin{array}{l} y + z = t + x + y \\ y + z = z + w \end{array}\right.$$

and because $z + w = z + (x + y + z) > y + z$ we must have that

$$y + z = t + x + y.$$

From the inscribed angle theorem we have that

$$\angle BAD = \frac{y+z}{2}.$$

Finally, notice that

$$\begin{aligned} 2\pi &= (x + y + t) + (z + w) \\ &= (y + z) + (z + (x + y + z)) \\ &= (y + z) + (z + y + y + z) \\ &= 3(y + z) \\ \therefore \frac{y+z}{2} &= \frac{\pi}{3} \\ \therefore \angle BAD &= 60^\circ \end{aligned}$$

as desired.

Problem 10. Let n be a positive integer. Alice receives a bag containing n distinct positive integers and writes on a board all possible numbers of the form $xy + z$, where x, y and z are (not necessarily distinct) numbers from the bag. Given n, determine the minimum number of *distinct* numbers that Alice must have written on the board, regardless of which numbers were in the bag.

Solution. For a set S of n positive integers, let

$$T(S) = \{xy + z \mid x, y, z \in S\}.$$

We claim that the smallest possible size of $T(S)$ is $n^2 + n - 1$.

Lemma 1. If $S = \{1, \ldots, n\}$ then $|T(S)| = n^2 + n - 1$.

Proof of Lemma 1. If $S = \{1, \ldots, n\}$ then it is clear that $T(S) = \{2, \ldots, n^2 + n\}$. Thus we have that $|T(S)| = n^2 + n - 1$, which was what we wanted to prove. □

Lemma 2. For all sets S of n positive integers it holds that $|T(S)| \geq n^2 + n - 1$.

Proof of Lemma 2. Let e be the smallest element of S and let E be the largest element of s.

We claim that all numbers of the forms

(i) $xE + z$, where $x, z \in S$, and

(ii) $e^2 + z$, where $z \in S \backslash \{E\}$

are pairwise distinct. This claim can be shown by considering the following cases:

- If $x_1E + z_1 = x_2E + z_2$ we get that $z_1 = z_2$ by considering the equation modulo E. Thereafter we immediately get that $(x_1, z_1) = (x_2, z_2)$.
- If $e^2 + z_1 = e^2 + z_2$ we must have that $z_1 = z_2$.
- Notice that $e^2 + z_1 < e^2 + E \leq eE + e \leq xE + z_2$ for all $x, z_2 \in S$, $z_1 \in S \backslash \{E\}$.

Since there are n^2 numbers of the form (i) and and $n - 1$ numbers of the form (ii) in $T(S)$, it follows that $|T(S)| \geq n^2 + n - 1$ for all sets S of n positive integers. □

Therefore the minimum number of distinct numbers that Alice may have written on the board is equal to $n^2 + n - 1$.

Problem 11. Emil has n stones whose weights are $1, 2, \ldots, n$ kilograms. Yesterday he attached a label to each stone showing its weight, but he is worried that his friend Ivar played a prank and swapped the labels during the night. Emil wants to determine whether the labels are correct or not by performing a number of weighings on his balance scale. After each weighing he is told which pan is heavier, or that both weigh the same. Can you come up with some weighings Emil can perform that are guaranteed to expose Ivar if he moved the labels?
You receive more points the fewer weighings your solution uses for large n!

Solution. As the problem description suggests, this is a fairly open-ended problem with multiple possible solutions. Here we present a very efficient method that requires only

$$2\lceil \log_2(n) \rceil + 1$$

weighings. We say that a stone is "determined" if Emil knows that the label on it is correct.

Start by weighing $1 + 2$ against 3, then $1 + 2 + 3$ against 6, then $1 + 2 + 3 + 6$ against 12, and so on. At each step, weigh all stones used so far against the stone that should weigh as much as their total. This way, the number on the stone on the right pan doubles in each step until the sum of all previously weighed stones exceeds n. The number on the last stone weighed will be $3 \cdot 2^{\lfloor \log_2(n/3) \rfloor}$. All labels on stones weighed so far — let's call this set X — can now be combined to form any integer weight between 1 and $6 \cdot 2^{\lfloor \log_2(n/3) \rfloor}$ (think of binary numbers, multiplied by three). In particular, if we haven't weighed we can pick some stones from X whose labeled weights sum to n and weigh them against the stone labeled n, and we do just that. In total, we have now used at most

$$\lfloor \log(\frac{n}{3}) \rfloor + 2$$

weighings. If any of the weighings result in the two pans being unequal, Emil immediately knows that some labels have been swapped. The same logic holds throughout, so from now on we assume every weighing gives the expected result. If stones 1 and 2 are correctly labeled, then every other stone weighed so far (i.e., all of X and the one labeled n) must also be correct. If 1 and 2 are mislabeled, they must either have been swapped with each other or their combined weight is more than 3, which would make the total weight of all weighed stones exceed the labeled values — which is impossible. However, we still need to ensure that stones 1 and 2 aren't just swapped, which can easily be verified in a single weighing. So now we have determined the correct labels of all stones in X and the stone labeled n.

Let Y be the set of stones that are still undetermined. We now show how to determine the identity of each of these in $\lceil \log_2(n) \rceil$ steps. We use the fact that an integer is uniquely determined by its first $\lceil \log_2(n) \rceil$ binary digits. For each integer $k \leq \lceil \log_2(n) \rceil$, let L_k and U_k denote the set of stones in Y whose k-th binary digit is 0 and 1, respectively. The rest of the algorithm is best described in pseudocode:

for $k = 1, \ldots, \lceil \log_2 n \rceil$: **do**
  if $\text{sum}(L_k) < \text{sum}(U_k)$: **then**
    Place all stones in L_k beside pan 1 (Don't put them on there just yet)
    while total weight beside pan 1 $>$ pan 2: **do**
      Put a stone from U_k beside pan 2, starting with the one with the **highest** label.
    end while
    Let D be how much more the stones besides pan 2 will weigh than those beside pan 1 if all labels are correct. Note: $D < n$.
    Pick stones from X with total weight D and put them beside pan 1.
    Perform the weighing!
  else
    Place all stones in U_k beside pan 1 (Don't put them on the pan just yet)
    while total weight beside pan 2 $>$ pan 1: **do**
      Put a stone from L_k beside pan 1, starting with the one with the **highest** label.
    end while
    Let D be how much more the stones besides pan 1 will weigh than those beside pan 2 if all labels are correct. Note: $D < n$.
    Pick stones from X with total weight D and put them beside pan 2.
    Perform the weighing!
  end if
end for

Claim: After k iterations of the algorithm, Emil has determined the first k binary digits of every stone's weight.

Proof of claim: By induction. Assume that after $k-1$ iterations, Emil knows the first $k-1$ binary digits of each stone's weight. Then in iteration k, pan 1 is as light as possible if the stones on it are correctly labeled, and pan 2 is as heavy as possible if the stones on it are correctly labled. Thus, if the set of stones in each pan is not as the lables suggest, then pan 1 would weigh more than pan 2.

If both pans weigh equally — which should happen if the labels are correct — then Emil knows which stones are in each pan. In Case 1, this means he knows exactly which are in L_k; in Case 2, he knows exactly which are in U_k. In either case, Emil now knows which stones in Y have a 0 and which have a 1 in their k-th binary digit, proving the claim. □

After k iterations, Emil knows the first k binary digits of each weight, and thus can verify whether the labels are correct. The total number of weighings used is:

$$\begin{aligned} &\lceil \log_2(\frac{n}{3}) \rceil + 2 + \lceil \log_2(n) \rceil \\ \leq &\lceil 2\log_2(n) \rceil + 1. \end{aligned}$$

This is not an optimal solution but it is impossible to solve the problem using fewer than $\lceil \log_3(n) \rceil$ weighings. The reason is that each weighing can divide the stones into at most three groups: those on pan 1, those on pan 2, and those not weighed. To ensure that no two labels are swapped, every pair of stones must appear in different groups in at least one weighing, which requires at least $\lceil \log_3(n) \rceil$ weighings.