HÖJD PUNKTEN

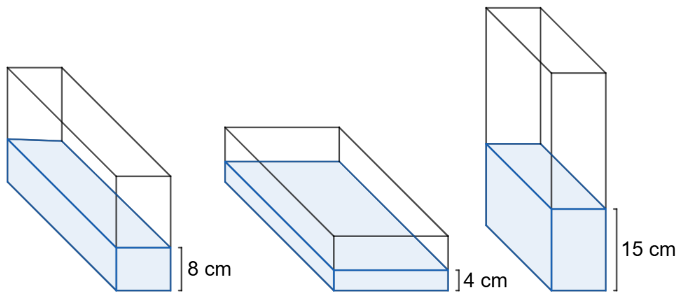
UVS MATEMATIKER

Gymnasietävling – 15 May 2025

Writing time: 3 hours
Allowed aids: only pen, eraser, compass and ruler
Each problem is worth 7 points.
For full points, justification is required unless otherwise stated.

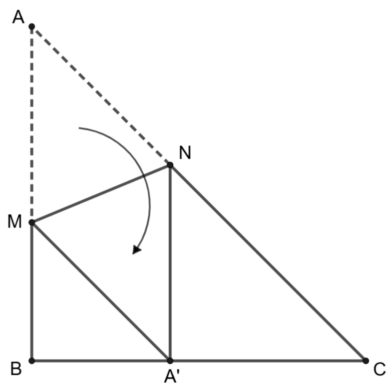
Problem 1. Sofia has a balance scale and nine weights weighing $10, 20, 40, 80, 160, 320, 640, 1280$, and 1410 grams, respectively. How can she balance these weights on the scale without any left over? Only an answer is required.

Problem 2. Cecilia pours three litres of water into a rectangular glass container with a lid. When she places it on a table, the water level becomes 8 cm, 4 cm or 15 cm depending on how she rotates it. What is the volume of the container?



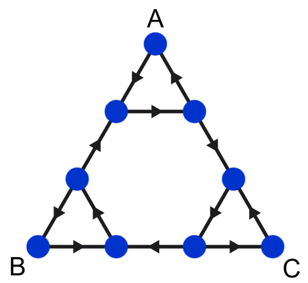
Problem 3. A palindromic number is a number that reads the same backwards as forwards, for example 494. If you multiply together all three–digit palindromic numbers, how many zeros does the product end with?

Problem 4. Theodor makes a triangle by halving a $1\,\text{dm} \times 1\,\text{dm}$ square sheet of paper along the diagonal. He then folds corner A to the point A' so that the crease MN is formed (where M and N lie on AB and AC, respectively) and so that $A'N$ is perpendicular to BC. In what ratio does A' divide the segment BC?



Problem 5. A frog is at the point $(0, 0)$ in the plane and starts jumping. Its first jump has length 1, and each subsequent jump is twice as long as the previous one. Every jump is made parallel to one of the coordinate axes. Which points can the frog reach by jumping in this way?

Problem 6. In a trivia game, the board looks like in the figure below. Three players, A, B, and C start in three different corners of the board. In each round they then take one step along an arrow. They are only allowed to move in the direction the arrow points. In the end all players have taken 20 steps. Prove that all players finish on different squares.



Problem 7. The Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ are the sequence obtained by starting with two ones and then computing each subsequent term by adding the two previous terms.

(a) Write the integers $1, 2, 3, \ldots, 20, 21$ in a circle in some order such that the difference between two adjacent numbers is always either 8 or 13.

(b) Let a, b and n be three consecutive Fibonacci numbers. Prove that it is possible to write the numbers $1, 2, \ldots, n$ in a circle so that the difference between two adjacent numbers is always either a or b.

Problem 8. The number $n = 10^9 + 7$ is a prime. The numbers $1, 2, 3, \ldots, n$ are placed on a circle in that order (so 1 is adjacent to n). A straight line is drawn that separates the numbers into two groups. Both groups have the same sum. Show that there is only one possible such division and find it.

Problem 9. Five points A, B, C, D and E lie on a circle in that order such that $|BA| = |BC|$, $|DA| = |DE|$ and $|BD| = |CE|$. Prove that $\angle BAD = 60°$.

Problem 10. Let n be a positive integer. Alice receives a bag containing n distinct positive integers and writes on a board all possible numbers of the form $xy + z$, where x, y and z are (not necessarily distinct) numbers from the bag. Given n, determine the minimum number of *distinct* numbers that Alice must have written on the board, regardless of which numbers were in the bag.

Problem 11. Emil has n stones whose weights are $1, 2, \ldots, n$ kilograms. Yesterday he attached a label to each stone showing its weight, but he is worried that his friend Ivar played a prank and swapped the labels during the night. Emil wants to determine whether the labels are correct or not by performing a number of weighings on his balance scale. After each weighing he is told which pan is heavier, or that both weigh the same. Can you come up with some weighings Emil can perform that are guaranteed to expose Ivar if he moved the labels?
You receive more points the fewer weighings your solution uses for large n!