# Geometry in Complex Numbers

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## 1 Introduction

Geometry is one of the few topics in Olympiad mathematics where a brute-force solution always exists. However, recognizing when such a solution is appropriate and being able to carry it out in reasonable time during a real contest requires training. Analytic geometry can be done in many ways, the most common choices of coordinate system being Cartesian, barycentric and complex. It turns out that for most Olympiad problems, complex numbers provide the most convenient computations as angles and directions can be directly incorporated in multiplication and conjugation. Many ideas from synthetic geometry are thus automatically baked into the algebra. Mostly, the reason for choosing to "bash" a geometry problem is to avoid hard and non-intuitive synthetic solutions, but in many cases complex numbers can yield both neat and elementary solutions. Below are some advantages and disadvantages of the method:

**Advantages**

- The unit circle. Many problem constructions are built upon a primary circle (e g the circumcircle of a triangle) which can be set as the unit circle.
- Midpoints. The arithmetic mean of two points is their midpoint.
- Special triangle centers. For a triangle on the unit circle, the expressions for many common triangle centers are very short.
- Angles and rotations.
- Parallelity and perpendicularity. As it happens, these conditions can be written entirely in terms of the points involved and their conjugates.
- Similar triangles. The properties of multiplication make the condition particularly pretty.

**Disadvantages**

- Intersections of lines. Apart from chords of the unit circle, intersecting lines in general is quite messy (it is no coincidence that a formula for the intersection of two lines is not included here).
- Multiple circles. Translating cyclic conditions in general into algebra will quickly lead to heavy expressions, often too unwieldy to be of use. Instead, it is better trying to use synthetic observations to simplify the calculations.
- Distances. Using the initial points as variables, it is only possible to find explicit expressions of squares of absolute values. Thus, in order to interpret distance conditions, all lengths must be squared first, which can be quite painful depending on the condition given.

Although reading solutions is a good way to learn the technique, it is still highly recommended to work on the problems yourself before reading the solutions. The problems are divided into three categories: warm-up, straightforward and advanced. The warm-up problems both have very short solutions. The straightforward problems are, as suggested by the name, straightforward and only involve standard procedures, although some can be relatively lengthy. The advanced problems are harder to approach directly and all require making some synthetic observations first and some careful planning before plunging into calculations, which for that matter are not necessarily long. In the following theorems and solutions, the coordinate of a point will be denoted by its lowercase letter unless otherwise specified.

# 2 Fundamentals

## 2.1 Arithmetics

- Complex numbers can be represented as vectors in the complex plane. A complex number can either be defined by its real and imaginary components or by its magnitude and argument.
- Addition between two complex numbers is equivalent to adding their real and imaginary components separately. This works in the same way as ordinary vector addition. In particular, the vector $\overrightarrow{AB}$ is given by $b - a$.
- Multiplication between two complex numbers is equivalent to multiplying their magnitudes and adding their arguments (for division, dividing and subtracting). This is the main advantage of working in complex numbers as angles are practically impossible to represent in usable form in Cartesian coordinates.
- The conjugate of a complex number is its reflection in the real axis; $z \in \mathbb{C}$ is real if and only if $z = \overline{z}$. If $w, z \in \mathbb{C}$, conjugation is simply done by $\overline{w+z} = \overline{w} + \overline{z}$ and $\overline{w \cdot z} = \overline{w} \cdot \overline{z}$. Moreover, $|z|^2 = z \cdot \overline{z}$, and in particular $\overline{z} = \frac{1}{z}$ for all $z$ on the unit circle.

## 2.2 Basic Tools

- $AB \parallel CD$ if and only if $\dfrac{a-b}{\overline{a}-\overline{b}} = \dfrac{c-d}{\overline{c}-\overline{d}}$.
- $AB \perp CD$ if and only if $\dfrac{a-b}{\overline{a}-\overline{b}} = -\dfrac{c-d}{\overline{c}-\overline{d}}$.
- In directed angles modulo 180°, $\measuredangle ABC = \measuredangle A'B'C'$ if and only if $\dfrac{a-b}{c-b} \bigg/ \dfrac{a'-b'}{c'-b'} \in \mathbb{R}$.
- $\triangle ABC \sim \triangle A'B'C'$ in the same orientation if and only if $\dfrac{a-b}{c-b} = \dfrac{a'-b'}{c'-b'}$; for opposite orientation, $\dfrac{a-b}{c-b} = \dfrac{\overline{a'}-\overline{b'}}{\overline{c'}-\overline{b'}}$.
- For $\triangle ABC$ on the unit circle, its orthocenter $H$, nine-point center $N$ and its centroid $G$ are given by $h = a + b + c$, $n = \dfrac{a+b+c}{2}$ and $g = \dfrac{a+b+c}{3}$.

Since taking the square root of a complex number is not a completely well-defined operation, problems involving midpoints of arcs can be handled using the following fact:

- If $\triangle ABC$ belongs to the unit circle, there exist $a$, $b$, $c$ such that $A$, $B$, $C$ are given by $a^2$, $b^2$, $c^2$ and the midpoints of arcs $BC$, $CA$, $AB$ not containing $A$, $B$, $C$ by $-bc$, $-ca$, $-ab$, respectively. Furthermore, the incenter $I$ of $\triangle ABC$ is given by $i = -ab - bc - ca$ and the $A$-excenter $I_a$ by $i_a = ab - bc + ca$ (with cyclic permutations).

### 2.3 Shortcuts

- If $A$, $B$ are on the unit circle, $\dfrac{a-b}{\overline{a}-\overline{b}} = -ab$.

- $z$ belongs to the chord $AB$ of the unit circle if and only if $\overline{z} = \dfrac{a+b-z}{ab}$. The tangent at a point $C$ can be regarded as the chord $CC$.

- The orthogonal projection $Z'$ of an arbitrary point $Z$ onto the chord $AB$ of the unit circle is given by $z' = \dfrac{a+b+z-ab\overline{z}}{2}$.

- The intersection $Z$ of two chords $AB$ and $CD$ of the unit circle is given by $z = \dfrac{ab(c+d)-cd(a+b)}{ab-cd}$.

- In particular, the intersection $Z$ of the tangents to the unit circle at $A$ and $B$ is given by $z = \dfrac{2ab}{a+b}$.

## 3 Problems

### 3.1 Warm-Up Problems

1. For $\triangle ABC$, let $H$ be its orthocenter, $M$ the midpoint of $BC$ and $N$ the reflection of $A$ in the circumcenter of $\triangle ABC$. Prove that $M$ is the midpoint of $HN$.

2. (Balkan 1984) Let $ABCD$ be a cyclic quadrilateral, and let $H_a$, $H_b$, $H_c$, $H_d$ be the orthocenters of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, $\triangle ABC$, respectively. Prove that the quadrilaterals $H_aH_bH_cH_d$ and $ABCD$ are congruent.

### 3.2 Straightforward Problems

3. (Inscribed Angle Theorem) Let $A$, $B$, $X$, $Y$ be points on a circle such that $X$ and $Y$ are on the same arc $AB$. Prove that $\angle AXB = \angle AYB$.

4. (Symmedian Line) The tangents to the circumcircle of $\triangle ABC$ at $B$ and $C$ intersect at $N$. If $M$ is the midpoint of $BC$, prove that $\angle BAN = \angle CAM$.

5. (Simson Line) Points $A$, $B$, $C$, $D$ lie on a circle. Denote by $P$, $Q$, $R$ the orthogonal projections of $D$ on $BC$, $CA$, $AB$, respectively. Prove that $P$, $Q$, $R$ are collinear.

6. (Napoleon's Triangle) Equilateral triangles with centroids $A_1$, $B_1$, $C_1$ are constructed on the sides $BC$, $CA$, $AB$ on the exterior of the triangle $ABC$. Prove that $\triangle A_1B_1C_1$ is equilateral.

7. The incircle with center $I$ touches the sides $BC$, $CA$, $AB$ of $\triangle ABC$ at $R$, $S$, $T$, respectively. Let $M$ and $N$ denote the midpoints of $AR$ and $BC$. Prove that $M$, $I$, $N$ are collinear.

8. (Baltic Way 2012/11) Let $ABC$ be a triangle with $\angle A = 60^\circ$. The point $T$ lies inside the triangle in such a way that $\angle ATB = \angle BTC = \angle CTA = 120^\circ$. Let $M$ be the midpoint of $BC$. Prove that $TA + TB + TC = 2AM$.

9. (Sweden 2012/6) A circle is inscribed in a trapezoid $ABCD$, with $AB \parallel CD$, and touches the sides $AB$, $BC$, $CD$, $DA$ at $Q$, $R$, $S$, $P$, respectively. Prove that $AC$ and $BD$ intersect on $QS$.

### 3.3 Advanced Problems

10. (IMO 2014/4) Points $P$ and $Q$ lie on side $BC$ of acute-angled triangle $ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points $M$ and $N$ lie on lines $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$, and $Q$ the midpoint of $AN$. Prove that lines $BM$ and $CN$ intersect on the circumcircle of triangle $ABC$.

11. (IMO 2003/4) Let $ABCD$ be a cyclic quadrilateral. Let $P$, $Q$, $R$ be the feet of the perpendiculars from $D$ to the lines $BC$, $CA$, $AB$, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with $AC$.

12. (IMOSL 2013/G2) Let $\omega$ be the circumcircle of a triangle $ABC$. Denote by $M$ and $N$ the midpoints of the sides $AB$ and $AC$, respectively, and denote by $T$ the midpoint of the arc $BC$ of $\omega$ not containing $A$. The circumcircles of the triangles $AMT$ and $ANT$ intersect the perpendicular bisectors of $AC$ and $AB$ at points $X$ and $Y$, respectively; assume that $X$ and $Y$ lie inside the triangle $ABC$. The lines $MN$ and $XY$ intersect at $K$. Prove that $KA = KT$.

13. (IMO 2010/2) Let $I$ be the incenter of triangle $ABC$ and let $\Gamma$ be its circumcircle. Let the line $AI$ intersect $\Gamma$ again at D. Let E be a point on the arc $BDC$ and $F$ a point on the side $BC$ such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. Finally, let $G$ be the midpoint of the segment $IF$. Prove that the lines $DG$ and $EI$ intersect on $\Gamma$.

14. (IMOSL 2011/G4) Let $ABC$ be an acute triangle with circumcircle $\Omega$. Let $B_0$ be the midpoint of $AC$ and let $C_0$ be the midpoint of $AB$. Let $D$ be the foot of the altitude from $A$ and let $G$ be the centroid of the triangle $ABC$. Let $\omega$ be a circle through $B_0$ and $C_0$ that is tangent to the circle $\Omega$ at a point $X \neq A$. Prove that the points $D$, $G$ and $X$ are collinear.

15. (IMO 2013/3) Let the excircle of triangle $ABC$ opposite the vertex $A$ be tangent to the side $BC$ at the point $A_1$. Define the points $B_1$ on $CA$ and $C_1$ on $AB$ analogously, using the excircles opposite $B$ and $C$, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle $ABC$. Prove that triangle $ABC$ is right-angled.

16. (Iran 2014 TST 3/6) The incircle with incenter $I$ of a non-isosceles triangle $ABC$ touches the side $BC$ at $D$. Let $X$ be a point on arc $BC$ of $(ABC)$ not containing $A$, such that if $E$ and $F$ are the orthogonal projections of $X$ on $BI$ and $CI$ and $M$ is the midpoint of $EF$, we have $MB = MC$. Prove that $\angle BAD = \angle CAX$.

## 4 Solutions

**Problem 1.** *For $\triangle ABC$, let $H$ be its orthocenter, $M$ the midpoint of $BC$ and $N$ the reflection of $A$ in the circumcenter of $\triangle ABC$. Prove that $M$ is the midpoint of $HN$.*

*Solution.* Letting $(ABC)$ be the unit circle in the complex plane, we have $h = a+b+c$, $m = \frac{b+c}{2}$ and $n = -a$, so $\frac{h+n}{2} = \frac{a+b+c-a}{2} = \frac{b+c}{2} = m$, as desired.

*Comment.* Although a synthetic proof is not hard to find, complex numbers can provide an immediate solution.

**Problem 2** (Balkan 1984)**.** *Let $ABCD$ be a cyclic quadrilateral, and let $H_a$, $H_b$, $H_c$, $H_d$ be the orthocenters of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, $\triangle ABC$, respectively. Prove that the quadrilaterals $H_aH_bH_cH_d$ and $ABCD$ are congruent.*

*Solution.* Set $(ABCD)$ as the unit circle in the complex plane. As $h_a = b+c+d$, $h_b = c+d+a$, $h_c = d+a+b$, $h_d = a+b+c$, we have $h_b - h_a = a - b$ with cyclic permutations, implying that the vectors $H_aH_b$, $H_bH_c$, $H_cH_d$, $H_dH_a$ and $AB$, $BC$, $CD$, $DA$ are antiparallel and of the same magnitude, respectively. Hence the two quadrilaterals are congruent.

*Comment.* Involving four orthocenters, this problem can be messy to approach synthetically. How do you even prove that two quadrilaterals are congruent? Complex numbers, however, trivializes the problem beautifully.

**Problem 3** (Inscribed Angle Theorem)**.** *Let $A$, $B$, $X$, $Y$ be points on a circle such that $X$ and $Y$ are on the same arc $AB$. Prove that $\angle AXB = \angle AYB$.*

*Solution.* Let their common circle be the unit circle in the complex plane. The statement is equivalent to

$$R = \frac{b-x}{a-x} \Big/ \frac{b-y}{a-y} = \frac{(b-x)(a-y)}{(b-y)(a-x)} \in \mathbb{R}.$$

But since

$$\overline{R} = \frac{\left(\frac{1}{b} - \frac{1}{x}\right)\left(\frac{1}{a} - \frac{1}{y}\right)}{\left(\frac{1}{b} - \frac{1}{y}\right)\left(\frac{1}{a} - \frac{1}{x}\right)} = \frac{(b-x)(a-y)}{(b-y)(a-x)} = R,$$

we have $R \in \mathbb{R}$, as required.

*Comment.* Proving that an expression is real is almost always done by proving that it is equal to its conjugate. With $A$, $B$, $X$, $Y$ on the unit circle, taking their reciprocals finishes the proof immediately. Make sure that you see why the expression for $R$ in the form above is invariant under conjugation.

**Problem 4** (Symmedian Line)**.** *The tangents to the circumcircle of $\triangle ABC$ at $B$ and $C$ intersect at $N$. If $M$ is the midpoint of $BC$, prove that $\angle BAN = \angle CAM$.*

*Solution.* Let $(ABC)$ be the unit circle in the complex plane. We have $n = \frac{2bc}{b+c}$ and $m = \frac{b+c}{2}$. The problem statement is equivalent to

$$R = \frac{c-a}{m-a} \Big/ \frac{n-a}{b-a} = \frac{(c-a)(b-a)}{\left(\frac{b+c}{2} - a\right)\left(\frac{2bc}{b-c} - a\right)} = \frac{2(c-a)(b-a)(b+c)}{(b+c-2a)(2bc-ab-ac)} \in \mathbb{R}.$$

Conjugation gives

$$\overline{R} = \frac{2\left(\frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{b} - \frac{1}{a}\right)\left(\frac{1}{b} + \frac{1}{c}\right)}{\left(\frac{1}{b} + \frac{1}{c} - \frac{2}{a}\right)\left(\frac{2}{bc} - \frac{1}{ab} - \frac{1}{ac}\right)} = \frac{2(c-a)(b-a)(b+c)}{(b+c-2a)(2bc-ab-ac)},$$

so that $R \in \mathbb{R}$.

*Comment.* Basically the same procedure as above. Keep in mind to clear nested fractions before conjugating.

**Problem 5** (Simson Line)**.** *Points $A$, $B$, $C$, $D$ lie on a circle. Denote by $P$, $Q$, $R$ the orthogonal projections of $D$ on $BC$, $CA$, $AB$, respectively. Prove that $P$, $Q$, $R$ are collinear.*

*Solution.* Set $(ABC)$ as the unit circle in the complex plane. We have

$$p = \frac{b+c+d-\frac{bc}{d}}{2}, \quad q = \frac{a+c+d-\frac{ac}{d}}{2}, \quad r = \frac{a+b+d-\frac{ab}{d}}{2},$$

with

$$p-q = \frac{b-a-\frac{bc}{d}+\frac{ac}{d}}{2} = \frac{(c-d)(a-b)}{2d}, \quad \overline{p}-\overline{q} = \frac{\left(\frac{1}{c}-\frac{1}{d}\right)\left(\frac{1}{a}-\frac{1}{b}\right)}{\frac{2}{d}} = \frac{(c-d)(a-b)}{2abc}$$

and

$$r-q = \frac{b-c-\frac{ab}{d}+\frac{ac}{d}}{2} = \frac{(d-a)(b-c)}{2d}, \quad \overline{r}-\overline{q} = \frac{\left(\frac{1}{d}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{c}\right)}{\frac{2}{d}} = \frac{(d-a)(b-c)}{2abc}.$$

Hence,

$$\frac{p-q}{\overline{p}-\overline{q}} = \frac{abc}{d} = \frac{r-q}{\overline{r}-\overline{q}},$$

implying that $P$, $Q$, $R$ are collinear.

*Comment.* Nothing strange here. We immediately write down the expressions for $p$, $q$ and $r$ and make the relevant subtractions. A good rule of thumb is to always look for factorizations as they generally simplify manipulations. Generally, recognizing factorizations is key to finishing longer computations.

**Problem 6** (Napoleon's Triangle)**.** *Equilateral triangles with centroids $A_1$, $B_1$, $C_1$ are constructed on the sides $BC$, $CA$, $AB$ on the exterior of the triangle $ABC$. Prove that $\triangle A_1B_1C_1$ is equilateral.*

*Solution.* Note that $AB_1 = \frac{\sqrt{3}}{3}AC$, $CA_1 = \frac{\sqrt{3}}{3}CB$, $BC_1 = \frac{\sqrt{3}}{3}BA$ and $\angle CAB_1 = \angle BCA_1 = \angle ABC_1 = \frac{\pi}{6}$. Letting $\zeta = \frac{\sqrt{3}}{3}e^{i\pi/6} = \frac{1}{2} + \frac{i\sqrt{3}}{6}$, we can write

$$b_1 - a = \zeta(c-a), \quad c_1 - b = \zeta(a-b), \quad a_1 - c = \zeta(b-c),$$

so that

$$b_1 - c_1 = \zeta(b+c-2a) + a - b, \quad a_1 - c_1 = \zeta(2b-a-c) + c - b.$$

We shall prove that $b_1 - c_1 = e^{i\pi/3}(a_1 - c_1)$, whence cyclically permuting the variables will imply that $\triangle A_1B_1C_1$ is equilateral. We have

$$b_1 - c_1 = \left(\frac{1}{2} + \frac{i\sqrt{3}}{6}\right)(b + c - 2a) + a - b = \frac{1}{2}(c - b) + \frac{i\sqrt{3}}{6}(b + c) - \frac{i\sqrt{3}}{3}a,$$

and

$$e^{i\pi/3}(a_1 - c_1) = \frac{i\sqrt{3}}{3}(2b - a - c) + \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)(c - b) = \frac{1}{2}(c - b) + \frac{i\sqrt{3}}{6}(b + c) - \frac{i\sqrt{3}}{3}a.$$

Thus $b_1 - c_1 = e^{i\pi/3}(a_1 - c_1)$, which finishes the proof.

*Comment.* A somewhat different technique is used here, as we actually make use of some $a + bi$-representations. The argument of cyclic permutation to extrapolate one's calculations is standard (and legitimate) – you don't need to do it all over again.

**Problem 7.** *The incircle with center $I$ touches the sides $BC$, $CA$, $AB$ of $\triangle ABC$ at $R$, $S$, $T$, respectively. Let $M$ and $N$ denote the midpoints of $AR$ and $BC$. Prove that $M$, $I$, $N$ are collinear.*

*Solution.* Set the incircle as the unit circle in the complex plane, so that

$$i = 0, \quad a = \frac{2st}{s + t}, \quad b = \frac{2rt}{r + t}, \quad c = \frac{2rs}{r + s}.$$

We then have

$$m = \frac{b + c}{2} = \frac{rt}{r + t} + \frac{rs}{r + s} = \frac{r(rs + rt + 2st)}{(r + s)(r + t)}, \quad \overline{m} = \frac{\frac{1}{r}\left(\frac{1}{rs} + \frac{1}{rt} + \frac{2}{st}\right)}{\left(\frac{1}{r} + \frac{1}{s}\right)\left(\frac{1}{r} + \frac{1}{t}\right)} = \frac{s + t + 2r}{(r + s)(r + t)}$$

and

$$n = \frac{a + r}{2} = \frac{st}{s + t} + \frac{r}{2} = \frac{rs + rt + 2st}{2(s + t)}, \quad \overline{n} = \frac{\frac{1}{rs} + \frac{1}{rt} + \frac{2}{st}}{2\left(\frac{1}{s} + \frac{1}{t}\right)} = \frac{s + t + 2r}{2r(s + t)}.$$

Thus,

$$\frac{m}{\overline{m}} = \frac{r(rs + rt + 2st)}{(r + s)(r + t)} \frac{(r + s)(r + t)}{s + t + 2r} = \frac{r(rs + rt + 2st)}{s + t + 2r}$$

and

$$\frac{n}{\overline{n}} = \frac{rs + rt + 2st}{2(s + t)} \frac{2r(s + t)}{s + t + 2r} = \frac{r(rs + rt + 2st)}{s + t + 2r},$$

implying that $\frac{m}{\overline{m}} = \frac{n}{\overline{n}}$ so that $M$, $I$, $N$ are collinear.

*Comment.* Sometimes, setting the incircle instead of the circumcircle as the unit circle is more convenient. The contact points $R$, $S$, $T$ then become the "variables", and $A$, $B$, $C$ are given by the intersections of the tangents, which have fairly clean expressions. Conveniently $I$ becomes the origin with this setup, and the problem amounts to prove that vectors $m$ and $n$ are parallel.

**Problem 8** (Baltic Way 2012/11)**.** *Let $ABC$ be a triangle with $\angle A = 60°$. The point $T$ lies inside the triangle in such a way that $\angle ATB = \angle BTC = \angle CTA = 120°$. Let $M$ be the midpoint of $BC$. Prove that $TA + TB + TC = 2AM$.*

*Solution.* Let $AM = x$, $BM = y$, $CM = z$. Since $\angle TAB = 60^\circ - \angle TAC = \angle TCA$, $\angle TBA = 60^\circ - \angle TAB = \angle TAC$, we have $\triangle TBA \sim \triangle TAC$ so that $x^2 = yz$. Setting $T$ as the origin in the complex plane, we have WLOG $a = x$, $b = \zeta y$, $c = \zeta^2 z$, where $\zeta = e^{2i\pi/3}$ (with the properties $\overline{\zeta} = \zeta^2$, $\overline{\zeta^2} = \zeta$ and $\zeta + \zeta^2 = -1$). Now $m = \frac{\zeta y + \zeta^2 z}{2}$, so that

$$m - a = \frac{\zeta y + \zeta^2 z - 2x}{2}, \quad \overline{m} - \overline{a} = \frac{\zeta^2 y + \zeta z - 2x}{2},$$

which gives

$$\begin{aligned}|m-a|^2 = (m-a)(\overline{m}-\overline{a}) &= \frac{(\zeta y + \zeta^2 z - 2x)(\zeta^2 y + \zeta z - 2x)}{4} \\ &= \frac{y^2 + z^2 + 4x^2 + (\zeta + \zeta^2)(yz - 2xz - 2xy)}{4} \\ &= \frac{y^2 + z^2 + 4x^2 + 2xz + 2xy - yz}{4}.\end{aligned}$$

Using $x^2 = yz \iff -3x^2 + 3yz = 0$, this becomes

$$|m-a|^2 = \frac{x^2 + y^2 + z^2 + 2xy + 2yz + 2zx}{4} = \frac{(x+y+z)^2}{4},$$

so that $\frac{(TA+TB+TC)^2}{4} = AM^2 \iff TA + TB + TC = 2AM$, which was to be proved.

*Comment.* The technique is non-standard yet intuitive. With the point $T$ such that the lines $AT$, $BT$, $CT$ are obtained by $120^\circ$ rotation of each other, it is natural to select $T$ as the origin and use the distances and $\zeta$ as initial variables. In principle, finding explicit absolute values to express distances is almost always a dead end – it is much easier to express squares of distances. The addition by $-3x^2 + 3yz = 0$ in the end is not far-fetched at all if you do the calculations first without invoking the $\angle A = 60^\circ$ condition and then realize that there is some "correction" needed, namely $x^2 = yz$, which is easily proved.

**Problem 9** (Sweden 2012/6). *A circle is inscribed in a trapezoid $ABCD$, with $AB \parallel CD$, and touches the sides $AB$, $BC$, $CD$, $DA$ at $Q$, $R$, $S$, $P$, respectively. Prove that $AC$ and $BD$ intersect on $QS$.*

*Solution.* Choose the incircle to $ABCD$ as the unit circle in the complex plane, and WLOG let $q = 1$, $s = -1$. We then have

$$a = \frac{2p}{p+1}, \quad b = \frac{2r}{r+1}, \quad c = \frac{-2r}{r-1}, \quad d = \frac{-2p}{p-1}.$$

From here, it is easy to see that interchanging $P$ and $R$ will interchange the pairs $(A, C)$ and $(B, D)$. If $X = AC \cap QS$, it suffices to prove that the expression for $x$ is symmetric in $p$ and $r$.

As $QS$ is the real axis, $x \in \mathbb{R}$, so

$$\frac{x-a}{x-\overline{a}} = \frac{x-c}{x-\overline{c}} \iff x^2 - \overline{c}x - ax + a\overline{c} = x^2 - \overline{a}x - cx + \overline{a}c \iff x = \frac{a\overline{c} - \overline{a}c}{(a-\overline{a}) - (c-\overline{c})}.$$

We have $\overline{a} = \frac{2}{p+1}$ and $\overline{c} = \frac{2}{r-1}$. We compute

$$a\overline{c} = \frac{2p}{p+1}\frac{2}{r-1} = \frac{4p}{(p+1)(r-1)}, \quad \overline{a}c = \frac{2}{p+1}\frac{-2r}{r-1} = \frac{-4r}{(p+1)(r-1)}$$

and

$$a - \overline{a} = \frac{2p}{p+1} - \frac{2}{p+1} = \frac{2(p-1)}{p+1}, \quad c - \overline{c} = \frac{-2r}{r-1} - \frac{2}{r-1} = \frac{-2(r+1)}{r-1}.$$

Hence,

$$x = \frac{\frac{4p+4r}{(p+1)(r-1)}}{\frac{2(p-1)}{p+1} + \frac{2(r+1)}{r-1}} = \frac{2(r+p)}{(p+1)(r+1)+(p-1)(r-1)},$$

which is symmetric in $p$ and $r$, as desired.

*Comment.* Generally, it is easier not to set one of the initial points on the unit circle equal to 1 as homogeneous expressions are faster to work with and easier to spot errors in. However, as $Q$ and $S$ are antipodal on the incircle and we seek the intersection with $QS$, it is in this case more convenient to have $QS$ as the real axis. Note that proving an expression is symmetric in some variables is a common way to finish a complex bash; this is because the intersection of lines in general is not particularly pretty. Here, it is much easier to prove that $AC \cap QS = BD \cap QS$ than $AC \cap BD \in QS$.

**Problem 10** (IMO 2014/4)**.** *Points $P$ and $Q$ lie on side $BC$ of acute-angled triangle $ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points $M$ and $N$ lie on lines $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$, and $Q$ the midpoint of $AN$. Prove that lines $BM$ and $CN$ intersect on the circumcircle of triangle $ABC$.*

*Solution.* Set $(ABC)$ as the unit circle in the complex plane. We shall prove that $BM \cap (ABC) = CN \cap (ABC)$. As points $M$ and $N$ are interchanged when interchanging $B$ and $C$, it is sufficient to prove that the expression for $X = BM \cap (ABC)$ is symmetric in $b$ and $c$. From the angle conditions, $\triangle PBA \sim \triangle ABC$ in opposite orientation, whence

$$\frac{p-b}{a-b} = \frac{\overline{a}-\overline{b}}{\overline{c}-\overline{b}} \quad \Longrightarrow \quad p = \frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{c}-\frac{1}{b}}(a-b)+b = \frac{c(a-b)^2}{a(c-b)} + b.$$

Furthermore, $p = \frac{m+a}{2}$ so that $m = 2p - a = \frac{2c(a-b)^2}{a(c-b)} + 2b - a$. With $X \in (ABC)$, we have

$$\frac{m-b}{\overline{c}-\overline{b}} = \frac{x-b}{\overline{x}-\overline{b}} = -bx \quad \Longrightarrow \quad x = -\frac{1}{b}\frac{m-b}{\overline{m}-\overline{b}}.$$

We compute

$$m - b = \frac{2c(a-b)^2}{a(c-b)} + b - a = \frac{2c(a-b)^2 - a(a-b)(c-b)}{a(c-b)} = \frac{(a-b)(ab+ac-2bc)}{a(c-b)},$$

and

$$\overline{m} - \overline{b} = \frac{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{ab}+\frac{1}{ac}-\frac{2}{bc}\right)}{\frac{1}{a}\left(\frac{1}{c}-\frac{1}{b}\right)} = \frac{(b-a)(b+c-2a)}{ab(b-c)}.$$

Thus, we have

$$x = -\frac{1}{b}\frac{m-b}{\overline{m}-\overline{b}} = -\frac{1}{b}\frac{(a-b)(ab+ac-2bc)}{a(c-b)}\frac{ab(b-c)}{(b-a)(b+c-2a)} = -\frac{ab+ac-2bc}{b+c-2a},$$

which is symmetric in $b$ and $c$, as desired.

*Comment.* It is hard to emphasize enough how important it is to work synthetically on a problem first before deciding to bash it. Actually, once aware of the similarity (and the trick to prove that $x$ is symmetric in $b$ and $c$), the rest almost follows automatically. The final solution is surprisingly short and clean.

**Problem 11** (IMO 2003/4). *Let $ABCD$ be a cyclic quadrilateral. Let $P$, $Q$, $R$ be the feet of the perpendiculars from $D$ to the lines $BC$, $CA$, $AB$, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with $AC$.*

*Solution.* Set $(ABC)$ as the unit circle in the complex plane. By the angle bisector theorem, the bisectors of $\angle ABC$ and $\angle ADC$ intersecting on $AC$ is equivalent to $\frac{AB}{BC} = \frac{AD}{CD} \iff AB^2CD^2 = BC^2AD^2 \iff |a-b|^2|c-d|^2 = |b-c|^2|d-a|^2$. Noting that $|a-b|^2 = (a-b)\left(\frac{1}{a}-\frac{1}{b}\right) = -\frac{(a-b)^2}{ab}$ with cyclic permutations, this is in turn equivalent to

$$\frac{(a-b)^2}{ab}\frac{(c-d)^2}{cd} = \frac{(b-c)^2}{bc}\frac{(d-a)^2}{da} \quad\iff\quad (a-b)^2(c-d)^2 = (b-c)^2(d-a)^2.$$

To incorporate the other condition $PQ = QR$, we compute

$$p = \frac{b+c+d-\frac{bc}{d}}{2}, \quad q = \frac{a+c+d-\frac{ac}{d}}{2}, \quad r = \frac{a+b+d-\frac{ab}{d}}{2},$$

so that

$$p-q = \frac{b-a-\frac{bc}{d}+\frac{ac}{d}}{2}, \quad \overline{p}-\overline{q} = \frac{\left(\frac{1}{c}-\frac{1}{d}\right)\left(\frac{1}{a}-\frac{1}{b}\right)}{\frac{2}{d}} = \frac{(c-d)(a-b)}{2abc}.$$

Thus,

$$|p-q|^2 = (p-q)(\overline{p}-\overline{q}) = \frac{(c-d)^2(a-b)^2}{4abcd}.$$

Similarly, we have

$$r-q = \frac{b-c-\frac{ab}{d}+\frac{ac}{d}}{2} = \frac{(d-a)(b-c)}{2d}, \quad \overline{r}-\overline{q} = \frac{\left(\frac{1}{d}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{c}\right)}{\frac{2}{d}} = \frac{(d-a)(b-c)}{2abc},$$

so that

$$|r-q|^2 = \frac{(d-a)^2(b-c)^2}{4abcd}.$$

With $PQ^2 = QR^2$, this gives

$$\frac{(c-d)^2(a-b)^2}{4abcd} = \frac{(d-a)^2(b-c)^2}{4abcd} \quad\iff\quad (a-b)^2(c-d)^2 = (b-c)^2(d-a)^2.$$

Hence, $PQ = QR$ if and only if the angle bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC, which was to be proved.

*Comment.* As illustrated above, proving if-and-only-if statements in complex numbers can be done by showing that both sides are equivalent to the same algebraic equality. After writing down the condition for $PQ = QR$, one can see that it involves the side vectors of $ABCD$, justifying our use of the angle bisector theorem. It is worth remembering that $|a-b|^2 = -\frac{(a-b)^2}{ab}$ for $A$, $B$ on the unit circle.

**Problem 12** (IMOSL 2013/G2)**.** *Let $\omega$ be the circumcircle of a triangle $ABC$. Denote by $M$ and $N$ the midpoints of the sides $AB$ and $AC$, respectively, and denote by $T$ the midpoint of the arc $BC$ of $\omega$ not containing $A$. The circumcircles of the triangles $AMT$ and $ANT$ intersect the perpendicular bisectors of $AC$ and $AB$ at points $X$ and $Y$, respectively; assume that $X$ and $Y$ lie inside the triangle $ABC$. The lines $MN$ and $XY$ intersect at $K$. Prove that $KA = KT$.*

*Solution.* Let $O_1$, $O_2$ denote the circumcenters of $(ANT)$, $(AMT)$, respectively, and let $\ell$ be the perpendicular bisector of $AT$. Since $AT$ is a chord in $(ANT)$, $(AMT)$ and $\omega$, we have $O_1, O_2, O \in \ell$. The problem statement $KA = KT$ is equivalent to $K \in \ell$, and we shall prove this by showing that $X = X'$, $Y = Y'$, where $X'$ and $Y'$ are the reflections of $M$ and $N$ in $\ell$, respectively.

Set $\omega$ as the unit circle in the complex plane. Then there exist $a$, $b$, $c$ such that $A$, $B$, $C$ are given by $a^2$, $b^2$, $c^2$ and the midpoints of $BC$, $CA$, $AB$ not containing $A$, $B$, $C$ by $-bc$, $-ca$, $-ab$, respectively. Note that $ATX'M$ and $ATY'N$ become isosceles trapezoids so that $X' \in (AMT)$, $Y' \in (ANT)$. Using the fact that $\triangle MAT \sim \triangle X'TA$ in opposite orientation, we can write

$$\frac{x'-t}{a^2-t} = \frac{\overline{m}-\overline{a^2}}{\overline{t}-\overline{a^2}} \quad\Longrightarrow\quad x' = -\frac{a^2-t}{\overline{a^2}-\overline{t}}(\overline{m}-\overline{a^2})+t = -a^2bc\left(\frac{\frac{1}{a^2}+\frac{1}{b^2}}{2}-\frac{1}{a^2}\right)-bc = -\frac{c(a^2+b^2)}{2b}.$$

We now check that $X'$ lies on the perpendicular bisector of $AC$ or, equivalently, $X'N \perp AC$. As $n = \frac{a^2+c^2}{2}$, we have

$$x'-n = -\frac{c(a^2+b^2)}{2b} - \frac{a^2+c^2}{2} = -\frac{a^2c+b^2c+a^2b+bc^2}{2b} = -\frac{(a^2+bc)(b+c)}{2b}$$

and

$$\overline{x'}-\overline{n} = -\frac{\left(\frac{1}{a^2}+\frac{1}{bc}\right)\left(\frac{1}{b}+\frac{1}{c}\right)}{\frac{2}{b}} = -\frac{(a^2+bc)(b+c)}{2a^2bc^2}.$$

Thus,

$$\frac{x'-n}{\overline{x'}-\overline{n}} = \frac{(a^2+bc)(b+c)}{2b}\frac{2a^2bc^2}{(a^2+bc)(b+c)} = a^2c^2 = -\frac{a^2-c^2}{\overline{a^2}-\overline{c^2}},$$

implying that $X'$ lies on the perpendicular bisector of $Ac$. By switching variables, one easily obtains that $Y'$ lies on the perpendicular bisector of $AB$. Since $X' \in (AMT)$ and $Y' \in (ANT)$, we must have $X = X'$, $Y = Y'$. Hence, $K = MN \cap XY \in \ell$ so that $KA = KT$, which was to be proved.

*Comment.* Seeing that $M$ and $N$ should be the reflections of $M$ and $N$ in $\ell$ is easy. The hard part about this problem is actually how to set up the bash. It turns out that working backwards and introducing phantom points is a good idea in complex numbers, too. By working a little, we have actually reduced the problem to another one occurring in recent competitions, namely Baltic Way 2014/12.

**Problem 13** (IMO 2010/2). *Let $I$ be the incenter of triangle $ABC$ and let $\Gamma$ be its circumcircle. Let the line $AI$ intersect $\Gamma$ again at $D$. Let $E$ be a point on the arc $BDC$ and $F$ a point on the side $BC$ such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. Finally, let $G$ be the midpoint of the segment $IF$. Prove that the lines $DG$ and $EI$ intersect on $\Gamma$.*

*Solution.* Set $\Gamma$ as the unit circle in the complex plane. Then there exist $a$, $b$, $c$ such that $A$, $B$, $C$ are given by $a^2$, $b^2$, $c^2$ and the midpoints of arcs $BC$, $CA$, $AB$ not containing $A$, $B$, $C$ by $-bc$, $-ca$, $-ab$, respectively. As $i = -ab - bc - ca$, we can write $g = \frac{f+i}{2} = \frac{f - ab - bc - ca}{2}$. If $X = DG \cap \Gamma$ ($X \neq D$), we have

$$\frac{g-d}{\overline{g}-\overline{d}} = \frac{x-d}{\overline{x}-\overline{d}} = -xd \quad \implies \quad x = -\frac{1}{d}\frac{g-d}{\overline{g}-\overline{d}}.$$

Furthermore, $g - d = \frac{f-ab-bc-ca}{2} + bc = \frac{f-ab+bc-ca}{2}$; since $\overline{f} = \frac{b^2+c^2-f}{b^2c^2}$, conjugation gives

$$\overline{g} - \overline{d} = \frac{\frac{b^2+c^2-f}{b^2c^2} - \frac{1}{ab} + \frac{1}{bc} - \frac{1}{ca}}{2} = \frac{ab^2+ac^2+abc-b^2c-bc^2-af}{2ab^2c^2}.$$

Thus,

$$x = \frac{abc(f-ab+bc-ca)}{ab^2+ac^2+abc-b^2c-bc^2-af}.$$

Define $Y = EI \cap \Gamma$ ($Y \neq E$). We shall prove the problem statement by showing that $X = Y$. We get that

$$y = -\frac{1}{e}\frac{e-i}{\overline{e}-\overline{i}} = -\frac{1}{e}\frac{e+ab+bc+ca}{\frac{1}{e}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}} = \frac{-abc(e+ab+bc+ca)}{abc+e(a+b+c)}.$$

Let $F' = AF \cap \Gamma$ ($F \neq A$). As $BC \parallel EF'$, we have

$$\frac{b^2-c^2}{\overline{b^2}-\overline{c^2}} = \frac{e-f'}{\overline{e}-\overline{f'}} \quad \implies \quad e = \frac{b^2c^2}{f'},$$

and as

$$f' = -\frac{1}{a^2}\frac{f-a^2}{\overline{f}-\overline{a^2}} = \frac{b^2c^2(a^2-f)}{a^2b^2+a^2c^2-b^2c^2-a^2f},$$

we have

$$e = \frac{a^2b^2+a^2c^2-b^2c^-a^2f}{a^2-f}.$$

Hence,

$$\begin{aligned}
y &= \frac{-abc\left(\frac{a^2b^2+a^2c^2-b^2c^2-a^2f}{a^2-f}+ab+bc+ca\right)}{abc+\frac{a^2b^2+a^2c^2-b^2c^2-a^2f}{a^2-f}(a+b+c)} \\
&= \frac{-abc(a^2b^2+a^2c^2-b^2c^2+a^2(ab+bc+ca)-f(a^2+ab+bc+ca))}{abc(a^2-f)+(a^2b^2+a^2c^2-b^2c^2-a^2f)(a+b+c)} \\
&= \frac{abc(a+b)(a+c)(ab+ac-bc-f)}{(a+b)(a+c)(ab^2+ac^2+abc-b^2c-bc^2-af)} \\
&= \frac{-abc(f-ab+bc-ca)}{ab^2+ac^2+abc-b^2c-bc^2-af} \\
&= x,
\end{aligned}$$

as desired, implying that $DG \cap EI \in \Gamma$.

*Comment.* Introducing $F'$ is probably the easiest way of linking $E$ with $F$, as the angle condition is equivalent to arcs $BF'$ and $CE$ being equal. Expressing $x$ in terms of $f$ and $y$ in terms of $e$ are straightforward computations, and once it's done, it only remains to connect $e$ and $f$ via $f'$. For an IMO #2, the procedures are fairly standard, although one has to be careful when selecting initial variables.

**Problem 14** (IMOSL 2011/G4)**.** *Let $ABC$ be an acute triangle with circumcircle $\Omega$. Let $B_0$ be the midpoint of $AC$ and let $C_0$ be the midpoint of $AB$. Let $D$ be the foot of the altitude from $A$ and let $G$ be the centroid of the triangle $ABC$. Let $\omega$ be a circle through $B_0$ and $C_0$ that is tangent to the circle $\Omega$ at a point $X \neq A$. Prove that the points $D$, $G$ and $X$ are collinear.*

*Solution.* By homothety, it is easy to see that $(AB_0C_0)$ is tangent to $\Omega$ at $A$. We have by the radical axis theorem that the tangents to $\Omega$ at $A$ and $X$ and $B_0C_0$ are concurrent; call this radical center $T$. Let $S = AA \cap BC$, so that $T$ is the midpoint of $AS$ (as $T = AA \cap B_0C_0$). We have

$$\bar{s} = \frac{2a-s}{a^2} = \frac{b+c-s}{bc} \quad \Longrightarrow \quad 2abc - bcs = a^2(b+c) - a^2 s \quad \Longrightarrow \quad s = \frac{a^2(b+c)-2abc}{a^2-bc},$$

so that

$$t = \frac{a+s}{2} = \frac{a^2(b+c)-2abc+a^3-abc}{2(a^2-bc)} = \frac{a^2(a+b+c)-3abc}{2(a^2-bc)}$$

and

$$\bar{t} = \frac{\frac{1}{a^2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{3}{abc}}{2\left(\frac{1}{a^2}-\frac{1}{bc}\right)} = \frac{ab+bc+ca-3a^2}{2a(bc-a^2)}.$$

Furthermore, as $T$ lies on the tangent to $\Omega$ at $X$, we have

$$\bar{t} = \frac{2x-t}{x^2} \quad \Longleftrightarrow \quad x^2 + \frac{2}{\bar{t}}x + \frac{t}{\bar{t}} = 0.$$

Since $A$ and $X$ are the points satisfying the above equation, Vieta's formula gives

$$ax = \frac{t}{\bar{t}} = \frac{3a^2bc - a^3(a+b+c)}{ab+bc+ca-3a^2} \quad \Longrightarrow \quad x = \frac{3abc - a^2(a+b+c)}{ab+bc+ca-3a^2}.$$

It remains to prove that $D$, $G$, $X$ are collinear. Since $d = \frac{1}{2}\left(a+b+c-\frac{bc}{a}\right)$ and $g = \frac{a+b+c}{3}$,

$$d-g = \frac{3(a+b+c)-\frac{3bc}{a}-2(a+b+c)}{6} = \frac{a(a+b+c)-3bc}{6a}$$

and

$$\overline{d}-\overline{g} = \frac{\frac{1}{a}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{3}{bc}}{\frac{6}{a}} = \frac{ab+bc+ca-3a^2}{6abc},$$

so that

$$\frac{d-g}{\overline{d}-\overline{g}} = \frac{a(a+b+c)-3bc}{6a}\frac{6abc}{ab+bc+ca-3a^2} = \frac{bc\big(a(a+b+c)-3bc\big)}{ab+bc+ca-3a^2}.$$

On the other hand, we have

$$x-g = \frac{3abc-a^2(a+b+c)}{ab+bc+ca-3a^2} - \frac{a+b+c}{3} = \frac{9abc-(a+b+c)(ab+bc+ca)}{3(ab+bc+ca-3a^2)}$$

and

$$\overline{x}-\overline{g} = \frac{\frac{9}{abc}-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)}{3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}-\frac{3}{a^2}\right)} = \frac{9abc-(a+b+c)(ab+bc+ca)}{3bc\big(a(a+b+c)-3bc\big)},$$

so that

$$\frac{x-g}{\overline{x}-\overline{g}} = \frac{bc\big(a(a+b+c)-3bc\big)}{ab+bc+ca-3a^2}.$$

Hence $\frac{d-g}{\overline{d}-\overline{g}} = \frac{x-g}{\overline{x}-\overline{g}}$, implying that $D$, $G$, $X$ are collinear, which was to be proved.

*Comment.* Again, it requires some synthetic work to find the radical center $T$, from where we can get $X$ by Vieta. The Vieta trick is standard when dealing with non-unique intersections such as circle–circle, circle–line and circle–tangents intersections; one will run into problems if trying to solve quadratic equations conventionally as it is generally not allowed to directly take the square-root of a complex number. Furthermore, even if both solutions are known, there is often no easy way of determining which one is which. In this case, since we know that $a$ is one of the solutions, we can easily find the other solution $x$. Proving that $D$, $G$, $X$ are collinear is then straightforward.

**Problem 15** (IMO 2013/3)**.** *Let the excircle of triangle $ABC$ opposite the vertex $A$ be tangent to the side $BC$ at the point $A_1$. Define the points $B_1$ on $CA$ and $C_1$ on $AB$ analogously, using the excircles opposite $B$ and $C$, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle $ABC$. Prove that triangle $ABC$ is right-angled.*

*Solution.* Denote by $S$ the circumcenter of $\triangle A_1B_1C_1$ and let $I_a$, $I_b$, $I_c$ be the excenters opposite to $A$, $B$, $C$, respectively. We choose WLOG $AB \le AC$ and $S$ to lie on arc $AC$ not containing $B$. It is known that $SC_1 = SB_1$, $C_1B = B_1C = \frac{AB+AC-BC}{2}$ and $\angle SBC_1 = \angle SBA = \angle SCA = \angle SCB_1$, so $\triangle SC_1B \cong \triangle SB_1C$ (checking the conditions for SSA-congruence is easy). Hence $SB = SC$ and $S$ is the midpoint of arc $BC$ containing $A$.

Let $(ABC)$ be the unit circle in the complex plane; let the points $A$, $B$, $C$ and the midpoints of arcs $BC$, $CA$, $AB$ not containing $A$, $B$, $C$ be given by $a^2$, $b^2$, $c^2$, $-bc$, $-ca$, $-ab$, respectively.

In particular, it is known that $s = bc$. We have that $i_a = ab - bc + ca$ and $\overline{i_a} = \frac{b+c-a}{abc}$, so

$$a_1 = \frac{b^2 + c^2 + i_a - b^2c^2\overline{i_a}}{2} = \frac{b^2 + c^2 + a(b+c) - \frac{bc}{a}(b+c)}{2}, \quad b_1 = \frac{c^2 + a^2 + b(c+a) - \frac{ca}{b}(c+a)}{2},$$

where the latter follows by cyclic permutation. Thus, we have

$$\begin{aligned} a_1 - s &= \frac{b^2 + c^2 + ab + ac - 2bc - \frac{bc}{a}(b+c)}{2} = \frac{a(b-c)^2 + (b+c)(a^2 - bc)}{2a}, \\ \overline{a_1} - \overline{s} &= \frac{\frac{1}{a}\left(\frac{1}{b} - \frac{1}{c}\right)^2 + \left(\frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{a^2} - \frac{1}{bc}\right)}{\frac{2}{a}} = \frac{a(b-c)^2 - (b+c)(a^2 - bc)}{2ab^2c^2} \end{aligned}$$

and

$$\begin{aligned} b_1 - s &= \frac{c^2 + a^2 - bc + ab - \frac{ca}{b}(c+a)}{2} = \frac{(b-c)(a^2 - bc) - a(b^2 - c^2)}{2b}, \\ \overline{b_1} - \overline{s} &= \frac{\left(\frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{a^2} - \frac{1}{bc}\right) - \frac{1}{a}\left(\frac{1}{b^2} - \frac{1}{c^2}\right)}{\frac{2}{b}} = \frac{(b-c)(a^2 - bc) + a(b^2 - c^2)}{2a^2bc^2} \end{aligned}$$

Setting $SA_1 = SB_1 \iff 4a^2b^2c^2|a_1 - s|^2 = 4a^2b^2c^2|b_1 - s|^2$, we have, writing both sides as differences of squares,

$$\begin{aligned} a^2(b-c)^4 - (b+c)^2(a^2 - bc)^2 &= (b-c)^2(a^2 - bc)^2 - a^2(b^2 - c^2)^2 \\ a^2\Big((b-c)^4 + (b^2 - c^2)^2\Big) &= (a^2 - bc)^2\Big((b-c)^2 + (b+c)^2\Big) \\ \Big((b-c)^2 + (b+c)^2\Big)\Big(a^2(b-c)^2 - (a^2 - bc)^2\Big) &= 0 \\ 2(b^2 + c^2)(ab - ac + a^2 - bc)(ab - ac - a^2 + bc) &= 0 \\ (b^2 + c^2)(a+b)(a-c)(a+c)(b-a) &= 0 \\ (b^2 + c^2)(a^2 - b^2)(a^2 - c^2) &= 0. \end{aligned}$$

Since none of the factors $(a^2 - b^2)$, $(a^2 - c^2)$ are zero, as that would imply $A = B$ or $A = C$, respectively, we must have $b^2 + c^2 = 0$. Hence, the vertices $B$ and $C$ are antipodal on $(ABC)$ so that $\angle BAC = 90°$. This concludes the proof.

*Comment.* Even IMO #3's can be broken down the hard way. Needless to say, the synthetic observation that $S$ is the midpoint of arc $BC$ immediately made a finish by complex numbers accessible. The idea of proving that $a^2 + b^2 = 0$ by factoring an expression equal to zero is not that far-fetched, having in mind that the sought statement should follow directly from $SA_1 = SB_1$. There are also alternative ways that do not invoke the fact that $S$ is the midpoint of arc $BC$, which use representations in elementary symmetric polynomials to be able to deal with the large amount of clutter produced.

**Problem 16** (Iran 2014 TST 3/6)**.** *The incircle with incenter $I$ of a non-isosceles triangle $ABC$ touches the side $BC$ at $D$. Let $X$ be a point on arc $BC$ of $(ABC)$ not containing $A$, such that if $E$ and $F$ are the orthogonal projections of $X$ on $BI$ and $CI$ and $M$ is the midpoint of $EF$, we have $MB = MC$. Prove that $\angle BAD = \angle CAX$.*

*Solution.* Let $O$ denote the circumcenter of $\triangle ABC$ and let $R$, $S$, $T$ denote the midpoints of arcs $BC$, $CA$, $AB$ not containing $A$, $B$, $C$, respectively. Setting $(ABC)$ as the unit circle in the complex plane, there exist $a$, $b$, $c$ such that $A$, $B$, $C$, $R$, $S$, $T$ are given by $a^2$, $b^2$, $c^2$, $-bc$, $-ca$, $-ab$, respectively. Let $G = AD \cap (ABC)$ $(G \neq A)$ and let $Y$ be the point such that $Y \in (ABC)$, $YG \parallel BC$, where $\angle BAD = \angle CAY$. It is easy to see that there is a unique point on arc $BC$ not containing $A$ satisfying the properties of $X$ since $E$ and $F$, and thus $M$, move horizontally as $Y$ does (where $BC$ is the horizontal axis). We shall prove that the choice of $X = Y$ satisfies the sought properties.

Projecting $X$ onto $BS$ (that is, $BI$) and $CT$ (that is, $CI$), we have

$$e = \frac{b^2 - ac + x - \frac{-ab^2c}{x}}{2}, \quad f = \frac{c^2 - ab + x - \frac{abc^2}{c}}{2}.$$

Thus,

$$m = \frac{e+f}{2} = \frac{2x + \frac{abc(b+c)}{x} + b^2 + c^2 - a(b+c)}{4}$$

and

$$\overline{m} = \frac{\frac{2}{x} + x\frac{1}{abc}\left(\frac{1}{b} + \frac{1}{c}\right) + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a}\left(\frac{1}{b} + \frac{1}{c}\right)}{4} = \frac{2\frac{ab^2c^2}{x} + (b+c)x + a(b^2+c^2) - bc(b+c)}{4ab^2c^2}.$$

Furthermore, $MB = MC$ implies that $M \in OR$, whence $\frac{m}{\overline{m}} = \frac{r}{\overline{r}} = b^2c^2$. We have then

$$2\frac{ab^2c^2}{x} + (b+c)x + a(b^2+c^2) - bc(b+c) = 2ax + \frac{a^2bc(b+c)}{x} + a(b^2+c^2) - a^2(b+c),$$

or

$$(2a - b - c)x + abc(ab + ac - 2bc)\frac{1}{x} = (b+c)(a^2 - bc). \tag{1}$$

Next, we compute the expression for $y$. $D \in BC$ implies $\overline{d} = \frac{b^2 + c^2 - d}{b^2c^2}$, and $ID \perp BC$ implies

$$\frac{\overline{d} - \overline{i}}{d - i} = -\frac{\overline{b^2} - \overline{c^2}}{b^2 - c^2} = \frac{1}{b^2c^2} \quad \Longrightarrow \quad \overline{d} = \frac{d - i}{b^2c^2} + \overline{i},$$

so that

$$\frac{b^2 + c^2 - d}{b^2c^2} = \frac{d - i}{b^2c^2} + \overline{i},$$

which gives

$$\begin{aligned} d = \frac{b^2 + c^2 + i - b^2c^2\overline{i}}{2} &= \frac{b^2 + c^2 - ab - bc - ca + b^2c^2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right)}{2} \\ &= \frac{ab^2 + ac^2 - a^2b - a^2c + b^2c + bc^2}{2a}. \end{aligned}$$

Using the fact that $GA \parallel DA$ and $G \in (ABC)$, we have

$$-ga^2 = \frac{g - a^2}{\overline{g} - \overline{a^2}} = \frac{d - a^2}{\overline{d} - \overline{a^2}} \quad \Longrightarrow \quad g = -\frac{1}{a^2}\frac{d - a^2}{\overline{d} - \overline{a^2}}.$$

Due to the subtraction by $a^2$, we can now factorize

$$d - a^2 = \frac{ab^2 + ac^2 - a^2b - a^2c + b^2c + bc^2 - 2a^3}{2a} = \frac{(a+b)(a+c)(b+c-2a)}{2a}$$

and

$$\overline{d} - \overline{a^2} = \frac{\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{2}{a}\right)}{\frac{2}{a}} = \frac{(a+b)(a+c)(ab+ac-2bc)}{2a^2b^2c^2}.$$

We have then

$$g = -\frac{1}{a^2}\frac{d-a^2}{\overline{d}-\overline{a^2}} = -\frac{1}{a^2}\frac{(a+b)(a+c)(b+c-2a)}{2a}\frac{2a^2b^2c^2}{(a+b)(a+c)(ab+ac-2bc)} = \frac{b^2c^2(2a-b-c)}{a(ab+ac-2bc)}.$$

As $YG \parallel BC$, we have $yg = b^2c^2$ so that

$$y = \frac{b^2c^2}{g} = b^2c^2\frac{a(ab+ac-2bc)}{b^2c^2(2a-b-c)} = \frac{a(ab+ac-2bc)}{2a-b-c}.$$

Substituting $y = x$ in Equation 1, we have

$$\begin{aligned}&(2a-b-c)\frac{a(ab+ac-2bc)}{2a-b-c} + abc(ab+ac-2bc)\frac{2a-b-c}{a(ab+ac-2bc)}\\ &= a^2(b+c) - 2abc + 2abc - bc(b+c) = (b+c)(a^2-bc),\end{aligned}$$

as desired. Hence, due to uniqueness, $X = Y$ so that $\angle BAD = \angle CAX$, which was to be proved.

*Comment.* The nature of the problem is similar to that of Problem 13. Due to the uniqueness argument, it suffices to prove the converse direction of the statement. We set up a necessary condition for a point satisfying the properties of $X$, and at last we connected $d$ and $y$ by standard computations, after which controlling that $x = y$ satisfies the equation was easy.