HÖJDPUNKTEN 2024

Open Competition 9th of May 2024

UVS MATEMATIKER

Time: 3 hours
Allowed Materials: only pen/pencil, paper, compass and ruler.
Explain your solutions. An answer without explanation gives zero points unless stated otherwise.

Problem 1. Let n be a positive integer. Show that there exists a sequence $a_0, a_1, a_2, \dots, a_n$ of integers greater than 1 such that

$$a_0! \cdot a_1! \cdot a_2! \cdots a_{n-1}! = a_n!.$$

Problem 2. A finite set of positive integers $\{k_1, k_2, \dots, k_n\}$ is given. Show that there is a positive integer m such that the numbers $mk_1, mk_2, \dots, mk_n$ all have a different number of divisors.

Problem 3.

(a) Tilda has a rectangular piece of cardboard that is 1 meter wide and 16/9 meters long. Show that she can cut it into two pieces that can be reassembled to form a square.

(b) Show that there are infinitely many numbers x such that a $1\,\mathrm{m} \times x\,\mathrm{m}$ board can be sawn into two parts and assembled into a square.

Problem 4. A polynomial is called *reciproical* if whenever a is a root, $1/a$ is also a root. Let f be an irreducible polynomial of degree at least 2. Assume that f has a root $b \in \mathbb{C}$ with $|b| = 1$. Show that f is reciprocal and has even degree.

Problem 5. Ung Vetenskapssport has grown! We know have many active members and volunteers who want access to all of UVS's various channels to reach our fantastic members. But the more people who gain access, the greater the security risk; what if someone's account gets hacked? We now need your help to solve this problem!
There are n individuals who want access to UVS's accounts. We want to devise a system that guarantees that if $1 \leq m \leq n$ individuals want to log in together, they can do so, but if only $m-1$ individuals want to log in, there is not enough information to do so.
After lengthy discussions, we came up with the following proposal. We choose a password with M digits $x_1 x_2 \dots x_M$, and then reveal some subset of these digits to each person who wants access to our accounts.

(a) Is it possible to distribute the digits in such a way that the requirement above is met?

(b) If the answer is yes, what is the smallest M for which it is possible (expressed in terms of n and m)?

Problem 6. Let $\triangle ABC$ be a triangle with $|AB| < |AC|$. Let B' be the point on segment AC such that $|AB| = |AB'|$. Let D be a variable point such that $|AB| = |AD|$, distinct from B and B'. Let the circumcircle of $\triangle B'CD$ intersect line BC again a second time at E, distinct from C. Prove that as the point D varies, the line DE passes through a fixed point.

Problem 7. Let $\triangle ABC$ be a triangle with circumcircle Ω and incenter I. Let ω be the circle centered at A passing through I. Let ω intersect Ω at P and Q. Let X be the intersection point of lines PQ and BC. Denote by ω_P , ω_Q the circumcircles of $\triangle PXI$ and $\triangle QXI$, respectively. Let ω_P , ω_Q intersect line BC again at P', Q' different from X. Prove that lines PP' and QQ' concur on ω.

Problem 8. Kevin wants to simulate a die with n sides. He has at his disposal a p-sided die for each prime $p < n$. Since Kevin doesn't have unlimited time to roll dice, he wants to minimize the expected value of the number of die rolls. *Note: This problem is worth a total of 10 points.*

(a) Show that if $n = pq$ for two primes p, q (which may not necessarily be distinct), then Kevin can simulate an n-sided die with exactly 2 die rolls. [1 point]

(b) Show that he always needs at least 2 rolls, but he can achieve an expected value less than 3 for all $n \geq 12$. [2 points]

(c) Show that there are infinitely many numbers n that are not of the form pq for two primes p and q, such that Kevin can simulate an n-sided die with exactly 2 die rolls. [7 points]

Problem 9. Given an integer $d < n$ and a real number ε, we say that a set of d orthonormal vectors $v_1, ..., v_d \in \mathbb{R}^n$ is ε-*balanced* if

$$\forall j \in \{1, 2, ..., n\} : \quad \left| \sum_{i=1}^{d} v_{ij}^2 - \frac{d}{n} \right| \leq \varepsilon$$

Let $n = d + 1$, and assume that $v_1, ..., v_d \in \mathbb{R}^{d+1}$ is a set of orthonormal, ε-balanced vectors. Show that there exists a set of orthonormal vectors $w_1, w_2, ..., w_d \in \mathbb{R}^{d+1}$ that is 0-balanced and satisfies

$$\sum_{i,j} (v_{ij} - w_{ij})^2 < Cd\varepsilon$$

for some constant C independent of d and ε.

Problem 10. Ivar and Ravi live in a large haunted house consisting of n rooms. Each room has a number of doors leading to other rooms. In total, there are $n - 1$ doors, and it's possible to go from any room to any other room through a series of doors.

One day, the ghosts of the house decide to make all the doors one-way! Each door is colored red on one side and indigo on the other side, and to ensure that Ivar and Ravi can't stick together, the ghosts ensure that Ivar can only go through indigo-colored doors while Ravi can only go through red doors.

Since the ghosts don't want to be too mean, they gave Ivar and Ravi the opportunity to flip doors so that the colors switch places, but only according to certain rules. You can only flip a door if you are in a room that you cannot leave because all doors have the wrong color, and in that case, you must flip *all* doors in that room at once! Show that, regardless of how the ghosts colored the doors and regardless of which rooms Ivar and Ravi are in initially, Ivar and Ravi can walk around the house and flip the doors so that any color configuration is achieved.

Problem 11. Let k be a positive integer. Let S be an infinite set of points in the Euclidean plane such that all closed discs of radius 1 contain at most k points in S. Show that there exists a positive constant C independent of k and S such that:

(a) There is a circle of radius $r = 1 + \frac{C}{2^k}$ containing at most k points in S.

(b) There is a circle of radius $r = 1 + \frac{C}{k}$ containing at most k points in S.

Note: If you can show (b) there is no need to also provide a separate proof for (a). If you have some result that's stronger than (a) but weaker than (b), we might award points for that.