HÖJDPUNKTEN 2024

Gymnasietävling, 8-9th of May 2024

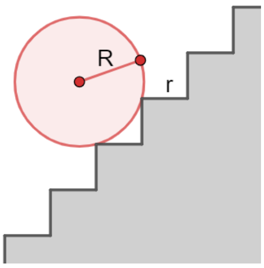
UVS MATEMATIKER

Time: 3 hours
Tools: Only pen, eraser, compass and ruler
Justify all your solutions. Only answers are not sufficient unless otherwise stated.

Problem 1. Mio has $3^{4^{5^2}}$ hotdogs and $9^{2^{7^2}}$ sausage bread. Is there enough sausage bread for all the hotdogs?
Note: power towers are calculated from top to bottom. 2^{2^3} *thus means* 2^8 *and not* 4^3.

Problem 2. Sebastian has 100 marbles in 10 different colors, but the number of marbles of each color may differ. He wants to place them in 10 piles with 10 marbles in each pile. Show that he can guarantee that there are at most two different colors in each pile.

Problem 3. A wheel of radius R is rolling up a staircase whose steps are squares of side length $r < R$. It is given that $R^2 = 2r^2$. How many steps does the wheel need to roll up before it has rotated a full revolution?



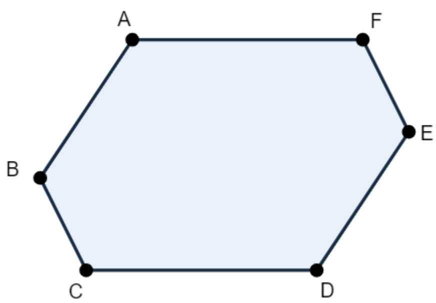
Problem 3

Problem 4. Show that there are infinitely many solutions to the equation $a!b! = c!$, where a, b and c are integers greater than or equal to 2.
Note: $n!$ *denotes the product of all positive integers less than or equal to* n.

Problem 5. What is the largest number whose digits are all different, and which is divisible by all its digits?

Problem 6. What is the integer part of the number $\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{19} + \frac{1}{20}$?

Problem 7. A convex hexagon $ABCDEF$ is given, where opposite sides are parallel. Show that the three diagonals AD, BE and CF intersect in a point if and only if the opposite sides are of equal length (ie $AB = DE$, $BC = EF$ and $CD = FA$).



Problem 7

Problem 8.

(a) Tilda has a rectangular piece of cardboard that is 1 meter wide and 2.25 meters long. Show that she can cut it into two pieces that can be reassembled to form a square.

(b) Show that there are infinitely many numbers x such that a $1\,\mathrm{m} \times x\,\mathrm{m}$ board can be sawn into two parts and assembled into a square.

Problem 9. Ruth is a cuboid-collector who only collects cuboids with volume n and integer side lengths. One day, she took out all her cuboids and lined them up. She then noticed a peculiar coincidence: when she looked at the blocks from above, no two of them appeared congruent. Furthermore, none of them was a square. Show that Ruth has no more than n cuboids.

Problem 10. Kevin has a $p-$sided die for every prime number p less than 42. Show that he can simluate a $42-$sided die (ie choose an integer between 1 and 42 so that the probability of getting any one of them is the same) using only:

(a) Three dice rolls.

(b) Two dice rolls.

Problem 11. Ivar and Ravi live in a large haunted house consisting of n rooms. Each room has a number of doors leading to other rooms. In total, there are $n - 1$ doors, and it's possible to go from any room to any other room through a series of doors.

One day, the ghosts of the house decide to make all the doors one-way! Each door is colored red on one side and indigo on the other side, and to ensure that Ivar and Ravi can't stick together, the ghosts ensure that Ivar can only go through indigo-colored doors while Ravi can only go through red doors.

Since the ghosts don't want to be too mean, they gave Ivar and Ravi the opportunity to flip doors so that the colors switch places, but only according to certain rules. You can only flip a door if you are in a room that you cannot leave because all doors have the wrong color, and in that case, you must flip *all* doors in that room at once! Show that, regardless of how the ghosts colored the doors and regardless of which rooms Ivar and Ravi are in initially, Ivar and Ravi can walk around the house and flip the doors so that any color configuration is achieved.