HÖJDPUNKTEN

Ung Vetenskapssport

Time: 3h

Tools: Only pen, eraser, compass and ruler

Öppen tävling, 11th of March 2023

Problem 1. Let M be the set of finite lists consisting only of the $-1s, 0s$ and $1s$. Say that a function $F : M \to \{-1, 0, 1\}$ is *amayzing* if it satisfies:

(a) $F(x) = F(y)$ for all x, y such that y is a permutation of x.

(b) $F(x) = -F(y)$ for all x, y such that $y = -x$ (i.e. $y_i = -x_i$ for all i).

(c) if $F(x) \in \{0, 1\}$ and we can get y by increasing some number in x, then $F(y) = 1$.

Determine all amayzing functions.

Problem 2. Sofia and her friends live in a city consisting of $n > 1$ parks, some pairs of which are connected by a street. It takes one minute of bike between any pair of parks connected by a street. Furthermore, it's possible to get between any pair of parks using the streets. Sofia's friends all live next to a park that is only connected to the rest of the parks by exactly one street. No two of her friends live next to the same park. Now Sofia wants to arrange a picnic in one of the parks, such that the total time it takes for all her friends to get there is at most $\frac{(n+1)^2}{8}$. Show that this is possible.

Problem 3. The polynomial $x^4 - 16x^3 + 88x^2 - 190x + 128$ has four positive roots. We draw a cyclic quadrilateral with these roots as side lengths (it is given that this is possible). What is the area of the quadrilateral?

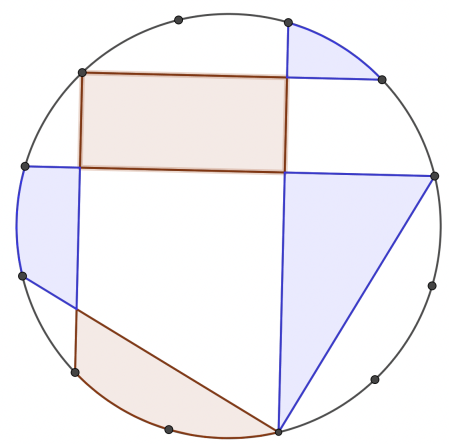
Problem 4. Determine all functions $f : \mathbb{Z}^+ \setminus \{1\} \to \mathbb{Z}^+$ such that

$$\phi(2mf(n)) = f(\phi(2n)m)$$

for all positive integers m och $n \neq 1$. Note that ϕ is Euler's totient function.

Problem 5. Let the integers be coloured with infinitely many colours. We say that a rational $(m \times n)-$matrix A is *interesting* if for every $i = 1, 2, ..., n$ there is a solution to $Ax = 0$ such that $x_i \neq 0$. Furthermore, we say that A is *good for the colour* c, if $Ax = 0$ has a solution $x \in \mathbb{Z}^n$ such that all x_i have the colour c. Is it possible that all intersting matrices are good for all (infinitely many) colours?

Problem 6. Prove that the sum of the blue areas is equal to the sum of the red areas. The figure is a circle and the points are evenly spaced.



Problem 7. Determine the smallest positive integer n such that if the numbers $404, 405, ..., n$ are divided into two groups, there are always three distinct numbers x, y, z that are in the same group such that $x + y = z$?

Problem 8. Say that a rational number is *nice* if it is of the form $\frac{k+1}{k}$ for some positive integer k. Given $n \in \mathbb{N}$, does there always exist a sequence of n rational numbers $q_1, ..., q_n$ such that $q_i q_{i+1} ... q_j$ is a nice number for all $1 \leq i \leq j \leq n$?

Problem 9. Låt $\alpha > 1$ be an irrational number and let n be a positive integer. Consider the set

$$S = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, ...\}.$$

Show that there exists an integer M such that S does not contain any arithmetic progression of length M with common difference n.

Problem 10. Let $\triangle PQR$ be a triangle, and let its incircle ω touch the sides in the points A, B and C respectively (where A is on PQ, B is on PR and C is on QR). Let X be the midpoint of the arc BC that doesn't contain A. Let the lines PX, QX intersect the lines AB, AC in the points M and N, respectively. Show that the circumcircle of AMN is tangent to ω.

Problem 11. Find all positive integer solutions to $m^{n+1} = 2^m + n^2$.

Problem 12. Is it true that for every finite group G there exists a subset of $\mathbb{R}^n$ (for some n) whose symmetry group is isomorphic to G?

Problem 13. In the land far, far away two teams are competing - the red team and the blue team. There are n cities, some pairs of which are connected by roads. In the beginning, the roads don't belong to anyone. The teams then take turns picking a road that has not yet been picked, and colour it with their own colour. The red team makes the first pick.
If at any point it's possible to travel between any pair of cities only using blue roads, the blue team wins. If all the roads have been picked (by some team) without the blue team achieving this, the red team wins.
Prove that the blue team wins if and only if it's possible to split the roads into two groups, such that within each group it's possible to travel between any pair of cities.

Problem 14. An integer n is given and the numbers $1, 2, 3, ..., n$ are written on the board. Kevin wants to pick k of them, and erase the rest, in such a way that no sum of some numbers left on the board is a perfect power. What is the largest k for which he can do this? *The organisers don't know the answer, and will give points for both upper and lower bounds. The better your bounds are asymptotically, the more points you get!*