HÖJDPUNKTEN
Ung Vetenskapssport

Open competition - solutions

Problem 1

Let M be the set of finite lists consisting only of the $-1s, 0s$ and $1s$. Say that a function $F : M \to \{-1, 0, 1\}$ is *amayzing* if it satisfies:

(a) $F(x) = F(y)$ for all x, y such that y is a permutation of x.

(b) $F(x) = -F(y)$ for all x, y such that $y = -x$ (i.e. $y_i = -x_i$ for all i).

(c) if $F(x) \in \{0, 1\}$ and we can get y by increasing some number in x, then $F(y) = 1$.

Determine all amayzing functions.

Solution

Note that:

- If x has the same number of $1s$ and $-1s$: $F(x) \overset{\text{(a)}}{=} F(-x) \overset{\text{(b)}}{=} -F(x) \implies F(x) = 0$
- If x has more $1s$ than $-1s$, there is some x_0 with the same number of $1s$ and $-1s$ (giving $F(x_0) = 0$ by above) for which we can increase some entries in x_0 to get x, giving that $F(x) = 1$ by (c)
- Finally if x has less $1s$ than $-1s$, (b) together with the previous observation gives $F(x) = -1$

Hence any amayzing F has to be

$$F(x) = \begin{cases} -1, & \text{if } x \text{ has more } -1\text{s than } 1\text{s} \\ 0, & \text{if } x \text{ the same number of } -1\text{s and } 1\text{s} \\ x, & \text{if } x \text{ has more } 1\text{s than } -1\text{s)} \end{cases}$$

and it's clear that all such F satisfy all the requirements.

Remark: This problem is essentially just about proving May's theorem.

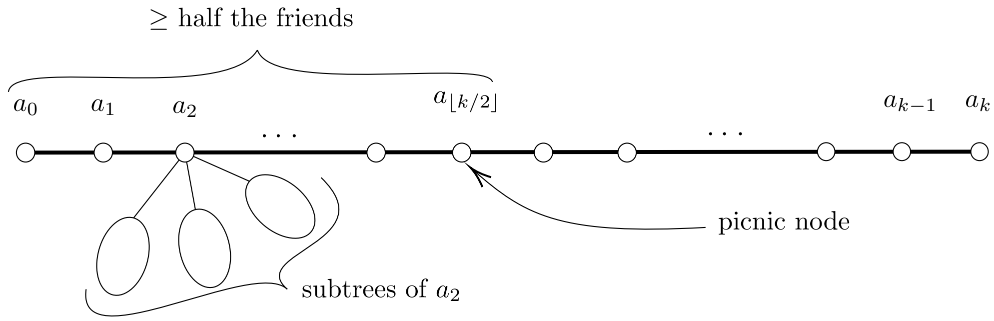
Problem 2

Sofia and her friends live in a city consisting of $n > 1$ parks, some pairs of which are connected by a street. It takes one minute of bike between any pair of parks connected by a street. Furthermore, it's possible to get between any pair of parks using the streets. Sofia's friends all live next to a park that is only connected to the rest of the parks by exactly one street. No two of her friends live next to the same park. Now Sofia wants to arrange a picnic in one of the parks, such that the total time it takes for all her friends to get there is at most $\frac{(n+1)^2}{8}$. Show that this is possible.

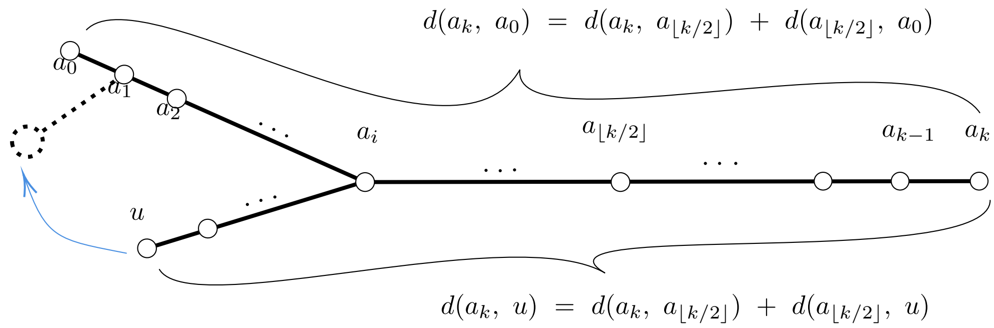
Solution 1

Consider the problem as a graph. Since having more edges can only decrease the travel time, it's enough to consider trees.

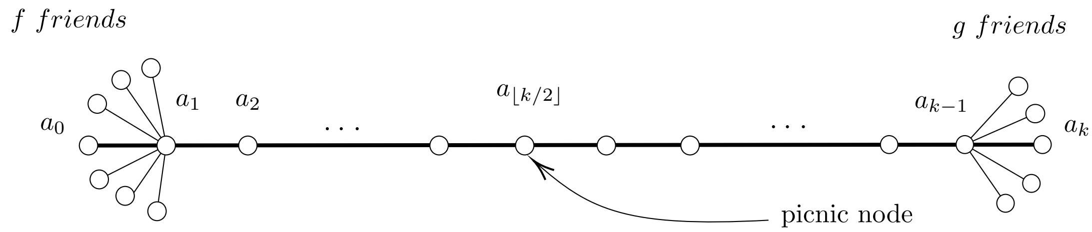
Assume that the longest path in the tree has length k, and assume that $a_0, ..., a_k$ is such a path. Removing all those nodes would split the remaining nodes into some number of smaller trees, each of which was originally attached to some a_i. We say that such a tree is a *subtree of* a_i. Assume wlog that the total number of friends in the subtrees of $a_0, ..., a_{\lfloor k/2 \rfloor}$ is at least as large as the corresponding quantity for $a_{\lceil k/2 \rceil}, ..., a_k$. Put the picnic in the node $a_{\lfloor k/2 \rfloor}$.



Consider a leaf node u (which contains a friend), other than a_0 or a_k, and assume it's in a subtree of a_i for some $i \leq \lfloor k/2 \rfloor$. The paths from a_k to both a_0 and u must then pass through $a_{\lfloor k/2 \rfloor}$. Since the distance from a_k to a_0 is maximal we hence get that $d(a_0, a_{\lfloor k/2 \rfloor}) \geq d(u, a_{\lfloor k/2 \rfloor})$ (where $d(x, y) =$ distance from x to y). So moving all such leaf nodes to instead be attached to a_1 will not decrease the sum of the distances from them to $a_{\lfloor k/2 \rfloor}$. For leaf nodes u which are in a subtree of a_i for some $i > \lfloor k/2 \rfloor$, we similarly get that $d(a_k, a_{\lfloor k/2 \rfloor}) \geq d(u, a_{\lfloor k/2 \rfloor})$, so moving them to instead be attached to a_{k-1} will not decrease the sum of the distances from them to $a_{\lfloor k/2 \rfloor}$.



We have now moved all nodes with friends so they are attached to a_1 or a_{k-1}, and want to show that the sum of the distances from them to $a_{\lfloor k/2 \rfloor}$ is at most $\frac{(n+1)^2}{8}$ (then we'll be done, as the distance has not decreased when we moved things around above). It is clear that we can assume that there are no other nodes than the leaves with friends and the nodes on the path $a_0, ..., a_k$, since that would just increase n without changing the travel time. In other words we have reduced the problem to just considering graphs which look as follows:



Note that we assumed (wlog) in the beginning that there were at least as many friends in the subtrees of $a_1, ..., a_{\lfloor k/2 \rfloor}$ as in the remaining subtrees. This now translates to there being at least as many leaves attached to a_1 as to a_{k-1}, i.e $f \geq g$ where f is the number of leaves attached to a_1 and g is the number of leaves attached to a_{k-1}. Since $a_{\lfloor k/2 \rfloor}$ is at least as close to a_1 as to a_{k-1}, we can move friends to attach to a_{k-1} instead, until the number of friends on each side is the same (or differ by at most 1 if the number of friends is odd), only making our situation worse. So we only need to consider the cases $g = f$ and $g = f - 1$. We now consider them separately.

Case 1: If $g = f$, we have $n = 2g + k - 1$ and the total distance from $a_{\lfloor k/2 \rfloor}$ to the leaves is

$$gk = \frac{1}{2} 2f(n - 2g + 1) \leq \frac{(n+1)^2}{8}$$

where we use AM-GM on $2g, n - 2g + 1$ in the last step.

Case 2: If $g = f - 1$, we have $n = 2g + k$ and the total distance from $a_{\lfloor k/2 \rfloor}$ to the leaves is

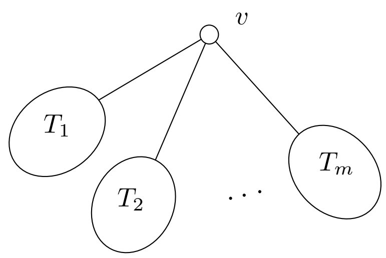
$$gk + \lfloor k/2 \rfloor \leq \frac{1}{2}(2g+1)k = \frac{1}{2}(2g+1)(n-2g) \leq \frac{(n+1)^2}{8}$$

where we use AM-GM on $2g + 1, n - 2g$ in the last step.

We are hence done.

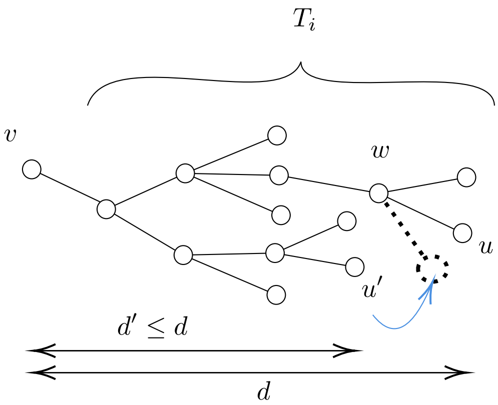
Solution 2

Consider the problem as a graph. Since having more edges can only decrease the travel time, it's enough to consider trees. Let the optimal node for the picnic be v, and root the tree there. Say that v has m subtrees $T_1, ..., T_m$. We want to show that the sum of the distances from v to all the friends is at most $\frac{(n+1)^2}{8}$. We will do this by keeping v fixed while moving the other nodes around a bit, only ever increasing the total distance to v from all the friends, until we have a graph for which it is easy to compute the total distance.

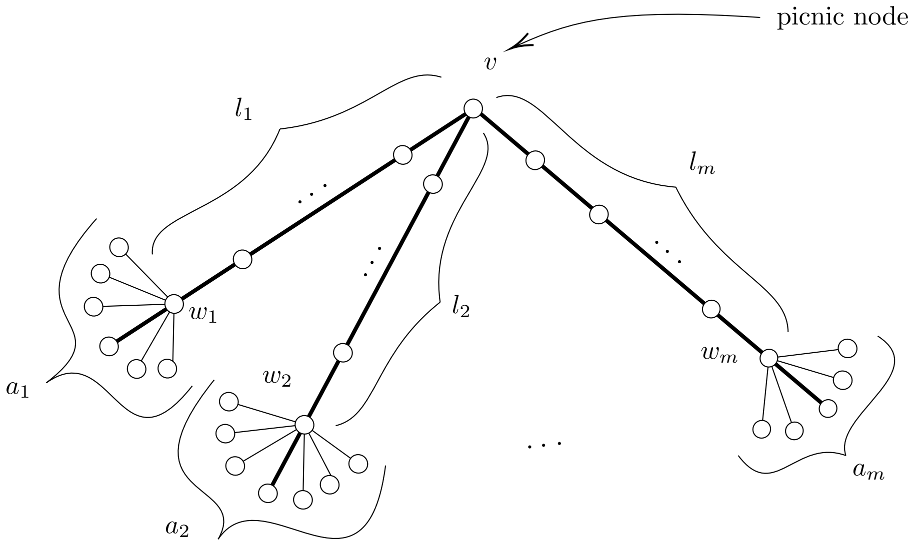


If any subtree T_i contained more than half of the friends, it would be strictly better to move the picnic one step towards the friends in that tree (as the travel distance would decrease by 1 for more than half the friends, and increase by 1 for less than half). Hence each T_i contains at most half of the friends, as v was picked to be an optimal node.

Consider the subtree T_i. Let d be largest distance from v to any node in T_i, and assume that the node u is at this distance from v. Clearly u is only connected to one node, say w, because otherwise there would be some node which is further from v. Now assume that there is some other leaf u' in T_i which is at distance $d' \leq d$ from v. Moving u' to instead be connected to w would cause it to be distance d from v, leaving all other distances to v unchanged. Hence if u' doesn't contain a friend, the sum of the distances from the friends to v will be unchanged, whereas if u' did contain a friend, it will increase by $d - d'$. If we can show that the sum of the distances to v in this modified graph is at most $\frac{(n+1)^2}{8}$, we will hence be done.



Repeating the above process will create a tree rooted at v where each of the subtrees T_i consists of a path of length l_i ending at a node w_i, with a_i leaves attached to w_i at the end, as in the picture below. Note that we only ever moved nodes within the subtrees, so the observation that each T_i contains at most half the friends is still valid. Also note that at this point, we may as well assume that every leaf contains a friend, as if we can do that case we can do every case.



We can further modify the tree by noting that given two subtrees T_i and T_j such that $l_i > l_j$, moving a leaf from T_j to T_i will increase the total travel distance to v by $l_i - l_j$ (as all leaves contain friends). If we assume wlog that T_1 and T_2 are the trees with the longest paths l_1, l_2, noting that they start out each containing at most half of the friends, we may move leaves from the other subtrees to these trees until they contain exactly $\left\lfloor \frac{f}{2} \right\rfloor$ leaves each, while only increasing the total travel time to v (where f is the total number of friends). Note that if f is even, this leaves no friends at the other subtrees, so we may remove those subtrees entirely (making n smaller and hence making it more difficult to prove the bound). If f is odd however, we might still have one friend left at the bottom of some other subtree. By making the path to that leftover friend shorter and the path in either T_1 or T_2 longer, we increase the total travel time, so in that case we may assume that the leftover friend is distance 1 from v. (*Note that we may have started out with some friends being in subtrees* *with paths of length* 0*, which corresponds to them being distance* 1 *from* v *- if this is the case we can think of it as* $\left\lfloor \frac{f}{2} \right\rfloor$ *of those nodes being the leaves of one subtree with a path of length* 0). We have now reduced the problem to the same two cases as in solution 1, so we are done by the same calculations that we performed there.

Problem 3

The polynomial $x^4 - 16x^3 + 88x^2 - 190x + 128$ has four positive roots. We draw a cyclic quadrilateral with these roots as side lengths (it is given that this is possible). What is the area of the quadrilateral?

Solution

Denote the polynmoial by $f(x)$. It's given that the polynomial has four positive roots, say a, b, c, d. By Vieta's formulas, we have $p = \frac{a+b+c+d}{2} = 8$. Hence by Brahmagupta's formula for the area of a cyclic quadrilateral, we get that the answer is

$$\sqrt{(p-a)(p-b)(p-c)(p-d)} = \sqrt{f(8)} = 12$$

Problem 4

Determine all functions $f : \mathbb{Z}^+ \setminus \{1\} \to \mathbb{Z}^+$ such that

$$\phi(2mf(n)) = f(\phi(2n)m)$$

for all positive integers m and $n \neq 1$. Note that ϕ is Euler's totient function.

Remark: Before starting the main solution, we will state some well-known facts without proof.

Fact 1: $\phi(mn) = \phi(m)\phi(n)$ *for all* m, n *which are coprime*

Fact 2: $\phi(n) = n \prod \left(1 - \frac{1}{p}\right)$ *where the product is over primes* p *dividing* n

Note that the second one follows from the first one, together with the fact that $\phi(p^\alpha) = (p-1)p^{\alpha-1}$ *for all primes* p *and* $\alpha \geq 1$ *(this is easy to check directly).*

Solution

We start by making two observations:

Obs 1 For all $n > 2$, $\phi(n)$ is even. This can be seen either by fact 2 above, or by noting that $k < n$ is coprime to n if and only if $n - k$ is coprime to n (and for $n > 2$ it's clear that $\frac{n}{2}$ is never coprime to n). In particular, if $n > 1$ is odd then $\phi(n)$ is even.

Obs 2 If n is divisible by p, we have that $\phi(pn) = p\phi(n)$. This follows from fact 2 above.

We claim that for every integer $\alpha \geq 0$

$$f(n) = \begin{cases} 2^\alpha \phi(n) & \text{if } n \text{ is even} \\ 2^{\alpha-1}\phi(n) & \text{if } n \text{ is odd} \end{cases}$$

is a solution, and that it's all of them. Note that $2^{\alpha-1}\phi(n)$ is an integer by observation 1.

Checking that these are solutions: We note that (using $\phi(n)$ divisible by 2 in the odd case)

$$\text{LHS} = \begin{cases} \phi(2^{\alpha+1}m\ \phi(n)) \overset{\text{(obs 2)}}{=} 2^\alpha \phi(2m\ \phi(n)) \overset{\text{(obs 2)}}{=} 2^\alpha \phi(m\ \phi(2n)) & \text{if } n \text{ is even} \\ \phi(2^\alpha m\ \phi(n)) \overset{\text{(obs 2)}}{=} 2^\alpha \phi(m\ \phi(n)) \overset{\text{(fact 1)}}{=} 2^\alpha \phi(m\ \phi(2n)) & \text{if } n \text{ is odd} \end{cases}$$

But this is exactly the same as the RHS, using that $\phi(2n)$ is even by observation 1.

Showing that these are all solutions: Let $f(2) = 2^\alpha c$, where c is odd. Put $m = 1, n = 2$:

$$\phi(2^{\alpha+1}c) = \phi(2f(2)) \overset{\text{(by assumption)}}{=} f(\phi(4)) = f(2) = 2^\alpha c \implies \phi(c) = c \implies c = 1$$

where we have used that c is odd, fact 1 and that $\phi(2^{\alpha+1}) = 2^\alpha$ in the first implication.

Put $n = 2 \implies f(2m) = \phi(2^{\alpha+1}m) \overset{\text{(obs 2)}}{=} 2^\alpha \phi(2m)$ using $f(2) = 2^\alpha$, giving the claim for even inputs.

For odd n, put $m = \phi(n)f(n)$, to get

$$\phi(2\phi(n)f(n)^2) = f(\phi(2n)\phi(n)f(n)) = f(\phi(n)^2 f(n))$$

where we used in the second step that $\phi(2n) = \phi(n)$ since n is odd. Since $n > 2$, observation 1 gives $\phi(n)$ even so the input to f on the RHS is even, so RHS is equal to $2^\alpha \phi(\phi(n)^2 f(n))$ using the formula for f for even inputs from above. Hence we have

$$\phi(2\phi(n)f(n)^2) = 2^\alpha \phi(\phi(n)^2 f(n))$$

Now observation 2 gives

$$\text{LHS} = 2f(n)\phi(\phi(n)f(n)) \qquad \text{RHS} = 2^\alpha \phi(n)\phi(\phi(n)f(n))$$

using that $2f(n) \mid \phi(n)f(n)$ for the LHS and $\phi(n) \mid \phi(n)$ for the RHS (noting again that $\phi(n)$ is even). We simplify to get

$$f(n) = 2^{\alpha-1}\phi(n)$$

for all odd n. So we are done.

Problem 5

Let the integers be coloured with infinitely many colours. We say that a rational $(m \times n)-$matrix A is *interesting* if for every $i = 1, 2, ..., n$ there is a solution to $Ax = 0$ such that $x_i \neq 0$. Furthermore, we say that A is *good for the colour* c, if $Ax = 0$ has a solution $x \in \mathbb{Z}^n$ such that all x_i have the colour c. Is it possible that all intersting matrices are good for all (infinitely many) colours?

Solution 1

The answer is yes. Let us construct a colouring with infinitely many colours for which all interesting matrices are good for all colours.

Given an interesting $(m \times n)-$matrix A, consider the kernel $K = \{v \in \mathbb{Q}^n \ : \ Av = 0\}$. We claim that K contains some vector $v \in \mathbb{Q}^n$ such that $v_i \neq 0$ for all i. Indeed, assume this was not the case, and pick a vector $v \in K$ such that $v_1, ..., v_{i-1} \neq 0$ but $v_i = 0$, where i is chosen to be the largest coordinate for which this is possible to achieve. Let $w \in K$ be such that $w_i \neq 0$ (this exists since A is interesting). Clearly $av + w \in K$ for all $a \in \mathbb{Q}$, and hence picking $a \notin \{-w_j/v_j : j < i\}$ we get that $(av + w)_j = av_j + w_j \neq 0$ for $j < i$. But also $(av + w)_i = w_i \neq 0$, so we arrived at a contradiction since we found a vector in K such that none of the coordinates $1, 2, ..., i$ are zero.

Let the number of colours be countably infinite. We want to show that the number of pairs of an interesting matrix and a colour is countable. Note that:

- There are countably many rational matrices of size $m \times n$ (each matrix has mn rational entries, so we can clearly inject the set into $\mathbb{Z}^{2mn}$ by sending each entry to a pair of integers, and then we can inject this set into $\mathbb{Q}$ - which is countable - for example by sending $(k_1, ..., k_{2mn}) \mapsto p_1^{k_1} \cdot ... \cdot p_{2mn}^{k_{2mn}}$ where $p_1, ..., p_{2mn}$ are distinct primes).
- The set of possible sizes of matrices is countable (similar reason to before).
- Combining the above two observations, we see that the number of rational matrices is a countable union of countable sets, which is hence countable (it clearly injects into $\mathbb{N}^2$ which is countable by similar reasoning to before). Hence there are clearly countably many interesting matrices, since that's a subset of all rational matrices.
- Finally, this gives that number of pairs of an interesting matrix and a colour is countable (similar reason to before).

Say that all such pairs of an interesting matrix and a colour are $(A_1, c_1), (A_2, c_2), ...$

Let us now inductively define a colouring of $\mathbb{Z}$ such that all interesting matrices are good for all colours. Assume that we have a partial colouring where finitely many numbers $a_1, ..., a_m$ have been coloured so far, such that we already know that A_i is good for c_i for all i up to $k - 1$. By above, A_k has some solution v such that $v_i \neq 0$ for all i. Since cv is a also a solution for all $c \in \mathbb{Z}$, we can assume that v has integer entries (by picking c to be the product of all denominators). Furthermore, by then picking $c > \max\{|a_1|, ..., |a_m|\}$, we can ensure that there is a solution cv to A_k with all entries having larger absolute value than any number coloured so far. But then we can extend the colouring by letting all those entries be coloured with c_k. This ensures that A_k is good for the colour c_k. Finally, we colour k and $-k$ with an arbitrary colour if they have not yet been coloured, to ensure that we eventually give every colour a number, before we continue the induction.

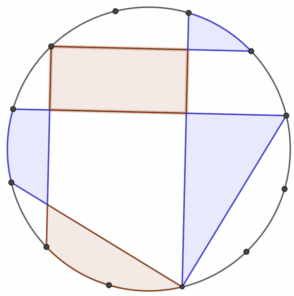
The above described process will define a colouring of all of $\mathbb{Z}$ for which every interesting matrix is good for every colour. Hence we are done.

Solution 2

Let $f(n)$ be the largest prime dividing $|n|$. Note that we can partition the primes into infinitely many infinite sets $P_1, P_2, ...$ (since there is a bijection between $\mathbb{N}^2$ and the primes). Now if $f(n) \in P_k$, we give n the colour k. As in solution 1, we know that every interesting matrix A has a solution v such that $v_i \neq 0$ for all i. Then for every $c \in \mathbb{Z}$, we have that cv also solves A. But then for all sufficiently large primes p, we can pick c such that cv has integer entries, and such that $f((cv)_i) = p$ for all i. Then the colour of all $(cv)_i$ are k, where $p \in P_k$. Since every P_k contains arbitrarily large primes, we can make sure that every entry of cv has colour k for any colour k. Hence we are done.

Problem 6

Prove that the sum of the blue areas is equal to the sum of the red areas. The figure is a circle and the points are evenly spaced.

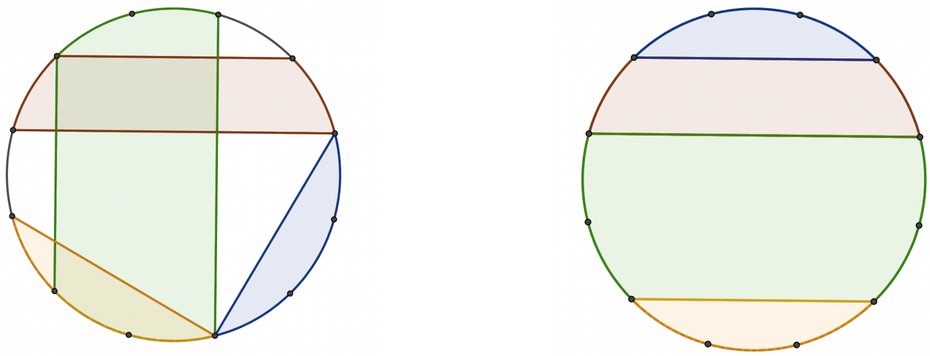


Solution 1

In the left figure below, four (overlapping) pieces of the circle have been coloured in four different colours, that have then been rearranged in the right figure to exactly cover the entire circle without overlap. Note that:

- red areas in the original figure (see problem statement above) correspond exactly to those areas where the pieces in the left figure below overlap
- blue areas in the original figure (see problem statement above) correspond exactly to those areas in the left figure below that are not covered at all

But the overlapping areas and the uncovered areas in the left figure must have the same area, since the 4 pieces together have the exact same area as the circle. Hence we are done.



Solution 2

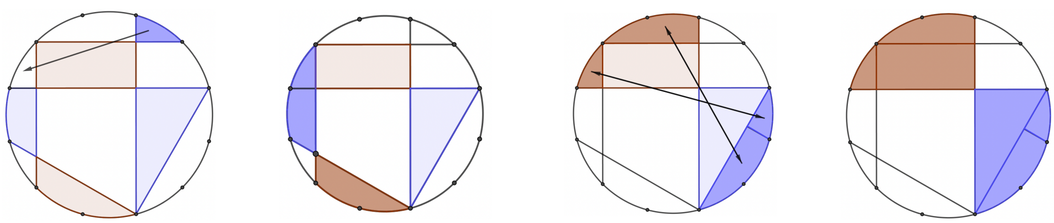
We get a different solution by "cutting and gluing", as in the four figures below. Since the red and the blue piece are the same size in the last picture, they must have been the same size from the start.

Fig 1 Move the dark blue piece in the top right along the arrow.

Fig 2 The dark red and dark blue pieces are the same size, so can be removed.

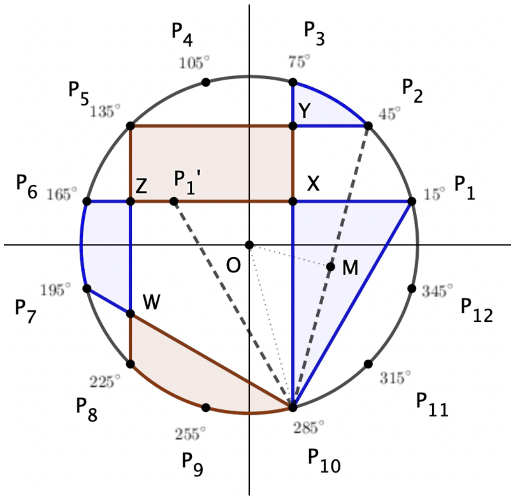
Fig 3 The dark red pieces are the same sizes are the dark blue pieces, so can add them all.

Fig 4 The dark red and dark blue piece are the same size, so can be removed.



Solution 3

It's also possible to solve the problem by simply computing the areas (or by combining some insights from solution 2 with computing the remaining areas). We will show here how to compute the area of the large blue triangle and the large red rectangle in two different ways (by expressing the coordinates with trigonometry, and by computing some lengths and angles), and hence show that they are the same size. Together with the observation in figure 1 och figure 2 from solution 2 this will give a complete solution, but it's also possible to use the methods here to compute all areas.



In the figure on the left the points have been named $P_1, ..., P_{12}$, the center of the circle O, and some of the intersections in the circle X, Y, Z and W. Furthermore we have placed the figure in a coordinate system where O is the origin, and the axes have been drawn such that they are parallel to P_1P_6 and P_4P_9 respectively. We may assume that the figure has been scaled such that the radius of the circle is 1, so it is the unit circle. The angle between OP_i and the $x-$axis has been written out for $P_1, ..., P_{12}$.

We have also constructed two additional points: P_1' is the reflection of P_1 in X, and M is the midpoint of P_2P_{10}.

Trigonometric solution: The coordinates for the points P_i and X, Y, Z are (all angles in degrees):

- $P_i : (\cos(30i - 15), \sin(30i - 15))$, by the definition of cosine and sine
- $X : (\cos(75), \sin(15))$ (the $x-$value is the same as for P_3, the $y-$value is the same as for P_1)
- $Y : (\cos(75), \sin(45))$ (the $x-$value is the same as for P_3 the $y-$value is the same as for P_2)
- $Z : (\cos(135), \sin(15))$ (the $x-$value is the same as for P_5, the $y-$value is the same as for P_1)

Hence the blue triangle has area (using that P_1X is perpendicular to $P_{10}X$)

$$\begin{aligned}\frac{|P_1X| \cdot |P_{10}X|}{2} &= \frac{(\cos(15) - \cos(75)) \cdot (\sin(15) - \sin(285))}{2}\\ &= \frac{(\cos(15) - \sin(15)) \cdot (\sin(15) + \cos(15))}{2}\\ &= \frac{\cos^2(15) - \sin^2(15)}{2}\\ &= \frac{1 - 2\sin^2(15)}{2} = \frac{1}{2} - \sin^2(15)\end{aligned}$$

where we used that $\sin(285) = -\cos(15)$ and that $\cos(75) = \sin(15)$ in the first step, factorisation of difference of two squares in the second step and the trigonometric formula $\sin^2(x) + \cos^2(x) = 1$ in the last step. The red rectangle has area (using that XZ is perpendicular to XY)

$$\begin{aligned}|XY| \cdot |XZ| &= (\sin(45) - \sin(15)) \cdot (\cos(75) - \cos(135))\\ &= (\frac{1}{\sqrt{2}} - \sin(15)) \cdot (\sin(15) + \frac{1}{\sqrt{2}})\\ &= \frac{1}{2} - \sin^2(15)\end{aligned}$$

where we used that $\cos(75) = \sin(15)$ and that $\sin(45) = \cos(45) = \frac{1}{\sqrt{2}}$ in the first step. Hence the red rectangle and the blue triangle have the same area. We can show that the remaining areas are also the same for red and blue (by similar computations or by the observations from solution 2).

Solution by computing lengths and angles: We start by computing the area of the blue triangle. Note that $\angle P_1OP_{10} = 90°$, since the arc from P_1 to P_{10} is exactly one fourth of the circle. Furthermore $|P_1O| = |P_{10}O| = 1$ since both are radii in the circle, so Pythagoras' theorem gives that $|P_1P_{10}| = \sqrt{1^2+1^2} = \sqrt{2}$. Note that

$$\angle XP_1P_{10} = \angle P_6P_1P_{10} = \frac{285° - 165°}{2} = 60°$$

by the inscribed angle theorem. This gives

$$|P_1X| = \frac{|P_1P_{10}|}{2} = \frac{1}{\sqrt{2}} \qquad\qquad |P_{10}X| = \sqrt{|P_1P_{10}|^2 - |P_1X|^2} = \sqrt{\frac{3}{2}}$$

using that P_1XP_{10} is a triangle with angles $30°, 60°, 90°$. Hence the area of the blue triangle is

$$\frac{|P_1X| \cdot |P_{10}X|}{2} = \frac{\sqrt{3}}{4}$$

We now compute the area of the red rectangle. Note that $|P_2Y| = |P_5Z| = |XY|$ by symmetry, say that they are all equal to t. Furthermore $\frac{|P_2P_{10}|}{2} = |P_2M| = \frac{\sqrt{3}}{2}$ since OMP_2 is a triangle with angles $30°, 60°, 90°$. We already know that $|P_{10}X| = \sqrt{\frac{3}{2}}$, so Pythagoras' theorem in the triangle P_2YP_{10} now gives that

$$\left(\sqrt{\frac{3}{2}} + t\right)^2 + t^2 = (\sqrt{3})^2 = 3 \quad\implies\quad 2t^2 + \sqrt{6}t + \frac{3}{2} = 3 \quad\implies\quad t = \frac{-\sqrt{3}+3}{2\sqrt{2}}$$

where we solve a quadratic in the last step. Furthermore $|P_2P_5| = |P_1P_{10}| = \sqrt{2}$ by symmetry (we computed $|P_1P_{10}|$ earlier), so we now get

$$|P_5Y| = |P_5P_2| - |P_2Y| = \sqrt{2} - \frac{-\sqrt{3}+3}{2\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

where we use that $|P_2Y| = t = \frac{-\sqrt{3}+3}{2\sqrt{2}}$. Hence the area of the red rectangle is

$$\begin{aligned} |P_5Y| \cdot |XY| &= \frac{1+\sqrt{3}}{2\sqrt{2}} \cdot \frac{-\sqrt{3}+3}{2\sqrt{2}} \\ &= \frac{-\sqrt{3}+3-3+3\sqrt{3}}{8} \\ &= \frac{\sqrt{3}}{4} \end{aligned}$$

So the blue triangle and the red rectangle have the same area. We can show that the remaining areas are also the same for red and blue (by similar computations or by the observations from solution 2).

Remark 1: These two solutions give us a way of computing $sin(15)$ since we now have two ways of expressing the area of the blue triangle:

$$\frac{1}{2} - sin^2(15) = \frac{\sqrt{3}}{4} \quad\implies\quad sin(15) = \frac{\sqrt{2-\sqrt{3}}}{2}$$

It is left as an exercise to the reader to also show that

$$sin(15) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Remark 2: For anyone who has not had enough ways to solve this problem, you are free to try showing that the blue triangle and the red rectangle have the same area by using that $|P_2Y| \cdot |P_5Y| = |P_3Y| \cdot |P_{10}Y|$ by the intersecting chords theorem, together with some observations about segments having the same lengths due to symmetry.

Problem 7

Determine the smallest positive integer n such that if the numbers $404, 405, ..., n$ are divided into two groups, there are always three distinct numbers x, y, z that are in the same group such that $x+y = z$?

Solution

The answer is 2023. In general we can ask the same question with 404 replaced by m. Then the answer is $n = 5m + 3$, which is what we will show here. We will refer to the two groups as the red group and the blue group, and think of it as colouring the numbers red and blue.

5m + 3 works:

Assume for contradiction that $5m+3$ doesn't work, i.e that it's possible to colour $m, m+1, ..., 5m+3$ red and blue such that there are *no* three different numbers x, y, z with the same colour such that $x + y = z$. This gives:

$$\text{If } x \neq y \text{ have the same colour, } x + y \text{ must have the other colour.} \qquad (*)$$
$$\text{If } x > y \text{ have the same colour, } x - y \text{ must have the other colour.} \qquad (**)$$

We'll show that these observations lead to a contradiction. Assume that m is red. Have four cases:

Fall 1 Assume that $m+1$ and $m+2$ are both red. Then $(*)$ gives that $2m+1 = m+(m+1)$ must be blue. By repeatedly applying $(*)$ in a similar way we get:

m	$m+1$	$m+2$	$2m+1$	$2m+2$	$3m+2$	$4m+3$	$5m+3$

Note that $(4m+3)-(m+1) = 3m+2 = (5m+3)-(2m+1)$. The first equality gives that $3m+2$ must be blue, by $(**)$. The second equality gives that $3m+2$ must be red, again by $(**)$. Contradiction.

Fall 2 Assume that $m+1$ is red and $m+2$ is blue. In a similar way to before we get from $(*)$ that:

m	$m+1$	$m+2$	$2m+1$	$2m+2$	$3m+3$	$4m+3$

Finally $(3m+3)-(m+1) = 2m+2$ gives that $2m+2$ is blue and $(4m+3)-(2m+1) = 2m+2$ gives that $2m+2$ is red, by $(**)$ in both cases. Contradiction.

Fall 3 Assume that $m+1$ is blue and $m+2$ is red. In a similar way to before we get from $(*)$ that:

m	$m+1$	$m+2$	$2m+1$	$2m+2$	$3m+3$	$4m+3$

Finally $(3m+3)-(m+2) = 2m+1$ gives that $2m+1$ is blue and $(4m+3)-(2m+2) = 2m+1$ gives that $2m+1$ is red, by $(**)$ in both cases. Contradiction.

Fall 4 Assume that $m+1$ and $m+2$ are both red. In a similar way to before we get from $(*)$ that:

m	$m+1$	$m+2$	$2m+1$	$2m+3$	$3m+3$	$4m+4$

Finally $(3m+3)-(m+2) = 2m+1$ gives that $2m+1$ is red and $(4m+4)-(2m+3) = 2m+1$ gives that $2m+1$ is blue, by $(**)$ in both cases. Contradiction.

In all four cases we derived a contradiction. Hence we have shown that $5m + 3$ works.

5m + 2 doesn't work:

Colour $m, m+1, ..., 2m$ and $4m+3, 4m+4, ..., 5m+2$ red, while $2m+1, 2m+2, ..., 4m+2$ are blue:

m	$m+1$	$\cdots$	$2m$	$2m+1$	$\cdots$	$4m+2$	$4m+3$	$\cdots$	$5m+1$	$5m+2$

If we pick two red numbers x and y, their sum $x + y$ can't be red, since:

- If x and y are both between m and $2m$, their sum is at least $2m + 1$ and at most $4m - 1$, so the sum is guaranteed to be blue.
- Otherwise at least one of x and y is larger than or equal to $4m + 3$, but then their sum is at least $5m + 3$, which is larger than all the numbers we care about.

If we pick two blue numbers x and y, their sum $x + y$ is at least $(2m + 1) + (2m + 2) = 4m + 3$, so it has to be red.

Hence it's possible to split the numbers $m, m+1, ..., 5m+2$ in two groups such that no three different numbers x, y, z with the same colour are such that $x + y = z$. So $5m + 2$ doesn't work.

Problem 8

Say that a rational number is *nice* if it is of the form $\frac{k+1}{k}$ for some positive integer k. Given $n \in \mathbb{N}$, does there always exist a sequence of n rational numbers $q_1, ..., q_n$ such that $q_i q_{i+1} ... q_j$ is a nice number for all $1 \leq i \leq j \leq n$?

Solution

The answer is yes. Let's guess that the numbers q_i are of the form

$$q_1 = \frac{k}{k-1}, \quad q_2 = \frac{k-1}{k-c_1}, ..., \quad q_{n-1} = \frac{k - c_1 ... c_{n-3}}{k - c_1 ... c_{n-2}}, \quad q_n = \frac{k - c_1 ... c_{n-2}}{k - c_1 ... c_{n-1}}$$

where $k, c_1, ..., c_{n-1}$ are positive integers and $c_i > 1$ for every i. What do we need for this to work?

- We need that for $i = 1$ and any $j = 1, 2, ..., n$, the number $q_1 ... q_j$ is nice, i.e. we need that

$$\frac{k}{k - c_1 ... c_{j-1}}$$

  is nice. This happens if and only if $c_1 ... c_{j-1} \mid k$ for all $j = 1, ..., n$. So letting $k = dc_1 ... c_{n-1}$ for some positive integer d, this will be true.

- We need that for $2 \leq i \leq j \leq n$, the number $q_i ... q_j$ is nice, i.e. we need that

$$\frac{k - c_1 ... c_{i-2}}{k - c_1 ... c_{j-1}} = \frac{k - c_1 ... c_{i-2}}{(k - c_1 ... c_{i-2}) + (c_1 ... c_{i-2} - c_1 ... c_{j-1})}$$

  is nice. This happens if and only if $c_1 ... c_{i-2} - c_1 ... c_{j-1} \mid k - c_1 ... c_{i-2}$, which (using that $k = dc_1 ... c_{n-1}$) is equivalent to

$$\begin{aligned}
& c_1 ... c_{i-2}(c_{i-1} ... c_{j-1} - 1) \mid c_1 ... c_{i-2}(dc_{i-1} ... c_{n-1} - 1) \\
\iff \quad & c_{i-1} ... c_{j-1} - 1 \mid dc_{i-1} ... c_{n-1} - 1 \\
\iff \quad & c_{i-1} ... c_{j-1} - 1 \mid (c_{i-1} ... c_{j-1} - 1) dc_j ... c_{n-1} + (dc_j ... c_{n-1} - 1) \\
\iff \quad & c_{i-1} ... c_{j-1} - 1 \mid dc_j ... c_{n-1} - 1
\end{aligned}$$

We have hence reduced the problem to showing that we can find positive integers $d, c_1, ..., c_{n-1} > 1$ such that for all $1 \leq i \leq j \leq n-1$, $c_i ... c_j - 1 \mid dc_{j+1} ... c_{n-1} - 1$ (where we shifted i, j by one here to simplify notation). We do this by induction on n.

$n = 2$: Only need $c_1 - 1 \mid d - 1$, so $c_1 = 2$ and any d works.

$n \to n+1$: Assume we have already picked $c_1, ..., c_{n-1}$ by induction. Then set

$$c_n = 1 + \prod_{1 \leq i \leq j \leq n-1} (c_i ... c_j - 1)$$

so that $c_n \equiv 1$ modulo $c_i ... c_j - 1$ for all $1 \leq i \leq j \leq n-1$. Note that $c_i \geq 2$ for $i = 1, 2, ..., n-1$, so get $c_n \geq 2$ as well. Now, for $1 \leq i \leq j \leq n$, considering $dc_{j+1} ... c_n - 1$ modulo $c_i ... c_j - 1$ we get that for any d

$$\begin{aligned}
dc_{j+1} ... c_n - 1 &\equiv dc_{j+1} ... c_{n-1} - 1 \\
&\equiv dc_{j+1} ... c_{n-2} - 1 \\
&... \\
&\equiv d - 1
\end{aligned}$$

using that $c_{j+1}, ..., c_n$ are all congruent to 1 mod $c_i ... c_j - 1$, by construction. Hence we just need to pick

$$d = 1 + \prod_{1 \leq i \leq j \leq n} (c_i ... c_j - 1)$$

and we are done.

Problem 9

Let $\alpha > 1$ be an irrational number and let n be a positive integer. Consider the set

$$S = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, ...\}.$$

Show that there exists an integer M such that S does not contain any arithmetic progression of length M with common difference n.

Solution 1

We start with a well-known lemma (known as Beatty's theorem).

Lemma (Beatty's Theorem). Given two irrational numbers α, β such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, the two sets $A = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, ...\}$ and $\{\lfloor \beta \rfloor, \lfloor 2\beta \rfloor, \lfloor 3\beta \rfloor, ...\}$ partition the natural numbers.
Proof: We need to prove that every natural number n is in either A and B, and that no n is in both A and B:

- If n is in both A and B, there exist k, m such that

  $$n < k\alpha < n+1 \qquad n < m\beta < n+1$$

  where the inequalities are strict since α, β are irrational. After dividing the inequalities by α and β respectively and summing, we get

  $$n < k + m < n+1$$

  using that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. This gives a contradiction, since there can't be an integer between n and $n+1$.

- If n is in neither A nor B, there exist k, m such that

  $$k\alpha < n < n+1 < (k+1)\alpha \qquad m\beta < n < n+1 < (m+1)\beta$$

  where the inequalities are strict since α, β irrational. After dividing the inequalities by α and β respectively and summing, we get

  $$k + m < n < n+1 < k + m + 2$$

  using that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. This gives a contradiction, since there can't be two integers between $k+m$ and $k+m+2$.

Let $\beta = \frac{\alpha}{\alpha - 1}$. Then the set $S' = \{\lfloor \beta \rfloor, \lfloor 2\beta \rfloor, \lfloor 3\beta \rfloor, ...\} = \mathbb{N} \setminus S$ by the lemma. Hence showing that there is an integer M for which S does not contain any arithmetic progression of length M with common difference n is equivalent to showing that for large enough M, every arithmetic sequence of length M and common difference n has some element in S'. We now prove this claim.

Consider $n, 2n, ..., \lceil \beta \rceil n$. By the pigeonhole principle, two of them are within distance 1 from each other modulo β (since there are $> \beta$ elements), i.e $\exists\ k, m \in \{1, ..., \lceil \beta \rceil\}$ and $\epsilon \in (0, 1)$ such that

$$kn - mn \equiv \epsilon \pmod{\beta} \quad \Longrightarrow \quad |k - m| = r \in \{1, 2, ..., \lceil \beta \rceil\} \ : \ rn \equiv \pm\epsilon \pmod{\beta}$$

Now, for any $a \in \mathbb{N}$, consider the arithmetic progression

$$a, a + rn, a + 2rn, ..., a + \left\lceil \frac{\beta}{\epsilon} \right\rceil rn$$

Since $rn \equiv \pm\epsilon \pmod{\beta}$, we are making "jumps" of size ϵ modulo β, and so after $\lceil \frac{\beta}{\epsilon} \rceil$ jumps we went one "lap" modulo β. But we also know that $|\epsilon| < 1$, and so after "one lap mod β" we must have hit a point $\delta \in (-1, 0)$ mod β, i.e we have some $s \in \{0, 1, ..., \lceil \frac{\beta}{\epsilon} \rceil\}$ such that

$$a + srn \equiv \delta \pmod{\beta} \quad \Longrightarrow \quad a + srn = k\beta + \delta \quad \text{for some } k \in \mathbb{N} \quad \Longrightarrow \quad a + srn = \lfloor k\beta \rfloor \in S'$$

But $sr \le \lceil \frac{\beta}{\epsilon} \rceil \cdot \lceil \beta \rceil$, where ϵ depends only on n (it's independent of a). Hence any arithmetic progression of length $> \lceil \frac{\beta}{\epsilon} \rceil \cdot \lceil \beta \rceil$ and common difference n contains an element in S'. So we are done.

Solution 2

It's also possible to prove it directly.

The main idea is similar to the end of the previous proof. For every $\epsilon > 0$, there must exist some $r \in \{1, 2, ..., \lceil \frac{\alpha}{\epsilon} \rceil\}$ and $\delta \in (0, \epsilon]$ such that $rn \equiv \pm\delta \pmod{\alpha}$, by the pigeonhole principle (similar reasoning to above).

Next consider the arithmetic progression $a, a + rn, a + 2rn, ..., a + \lceil \frac{\alpha}{\delta} \rceil rn$. Since $rn \equiv \pm\delta \pmod{\alpha}$, we are making "jumps" of size δ modulo α, and so after $\lceil \frac{\alpha}{\delta} \rceil$ jumps we went one "lap" modulo α. But we also know that $|\delta| < \epsilon$, and so after "one lap mod α" we must have hit a point $\gamma \in (0, \epsilon)$ mod α, i.e we have some $s \in \{0, 1, ..., \lceil \frac{\alpha}{\delta} \rceil\}$ such that

$$a + srn \equiv \gamma \pmod{\alpha} \quad \implies \quad a + srn = k\alpha + \gamma \quad \text{for some } k \in \mathbb{N}$$

If we pick $\epsilon < \alpha - 1$, we get that $\gamma \in (0, \alpha - 1)$, giving that

$$a + srn = k\alpha + \gamma < (k + 1)\alpha - 1 \quad \implies \quad \lfloor k\alpha \rfloor < a + srn < \lfloor (k + 1)\alpha \rfloor \quad \implies \quad a + srn \notin S$$

But $sr \leq \lceil \frac{\alpha}{\delta} \rceil \cdot \lceil \frac{\alpha}{\epsilon} \rceil$, where we can pick $\epsilon = \alpha - 1$ and δ then only depends on n and α (it's independent of a). Hence no arithmetic progression of length $> \lceil \frac{\alpha}{\delta} \rceil \cdot \lceil \frac{\alpha}{\alpha - 1} \rceil$ and common difference n is contained entirely in S.

Problem 10

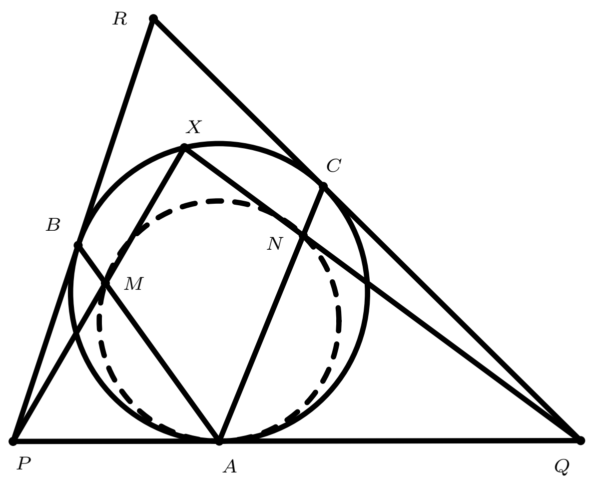
Let $\triangle PQR$ be a triangle, and let its incircle ω touch the sides in the points A, B and C respectively (where A is on PQ, B is on PR and C is on QR). Let X be the midpoint of the arc BC that doesn't contain A. Let the lines PX, QX intersect the lines AB, AC in the points M and N, respectively. Show that the circumcircle of AMN is tangent to ω.

Solution

We first note that the claim is equivalent to proving that there exists a homothety centered at A which takes the circle (AMN) to the circle (ABC). This is in turn equivalent to showing that $\frac{|BA|}{|MA|} = \frac{|CA|}{|NA|}$ which is equivalent to showing

$$\frac{|BM|}{|MA|} = \frac{|CN|}{|NA|}$$

using that $|BA| = |BM| + |MA|$ and $|CA| = |CN| + |NA|$.



Using part (b) of the lemma below for $\triangle XBA$ and $\triangle XCA$, we get

$$\frac{|BM|}{|MA|} = \left(\frac{|XB|}{|XA|}\right)^2 \quad \text{and} \quad \frac{|CN|}{|NA|} = \left(\frac{|XC|}{|XA|}\right)^2$$

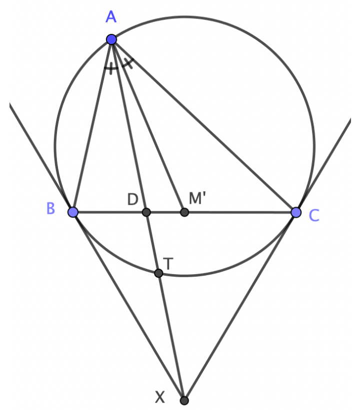
so using that $|XB| = |XC|$ (given in the problem statement), we are done!

It remains to show the following (well-known) lemma about symmedians:

Lemma: Given a triangle ABC, let X be the intersection between the tangents to (ABC) at B and C respectively. Let AX intersect BC in the point D. Then

(a) The line AX is the symmedian from A in the triangle ABC (meaning it's the reflection of the median in the angle bisector).

(b) We have that $\frac{|BD|}{|DC|} = \left(\frac{|AB|}{|AC|}\right)^2$.



Proof: Let T be the (second) intersection of AX with the circumcircle. Let M' be a point on BC such that AM' is the reflection of AX in the angle bisector from A. For (a), it's enough to show that M' is the midpoint of BC. Note that

$$\frac{|AM'|}{|M'C|} = \frac{|AB|}{|BT|} = \frac{|AX|}{|BX|} = \frac{|AX|}{|CX|} = \frac{|AC|}{|CT|} = \frac{|AM'|}{|M'B|}$$

where we have used (in order) that $\triangle ACM' \sim \triangle ATB$ (as $\angle CAM' = \angle TAB$ by construction of M' and $\angle ATB = \angle ACM'$ by inscribed angle theorem), $\triangle XBA \sim \triangle XTB$ (as $\angle XAB = \angle XBT$ since

XB is a tangent), $|BX| = |CX|$ (again since they are tangents) and then analogous statements to the first two steps in the last two steps. It follows that M' is the midpoint of BC, so we have proven (a).

Now (b) follows from:

$$\frac{|BD|}{|DC|} = \frac{|BD|}{|DC|} \cdot \frac{|BM'|}{|M'C|} = \frac{|BD|}{|M'C|} \cdot \frac{|BM'|}{|DC|} = \frac{|AB|/\sin(\angle BDA)}{|AC|/\sin(\angle CM'A)} \cdot \frac{|AB|/\sin(\angle BM'A)}{|AC|/\sin(\angle CDA)} = \left(\frac{|AB|}{|AC|}\right)^2$$

where in the first step we used that M' is the midpoint of BC, in the third step we used law of sines in the triangles $\triangle ABD, \triangle ACD, \triangle ABM'$ and $\triangle ACM'$ (together with noting that the angle from A to BD is the same as the angle from A to $M'C$ for the first fraction, and the analogous statement for BM' and DC for the second fraction), and in the final step we used that $\angle BDA + \angle CDA = 180°$ and $\angle BM'A + \angle CM'A = 180°$. $\square$

Remark: The lemma we used is a well-known lemma about symmedians. Note that property (b) is a special case of a property which more generally holds for isogonal lines, more precisely that if D and E lie on BC such that AD and AE are reflections of each other in the bisector from A, then

$$\frac{|BD|}{|CD|} \cdot \frac{|BE|}{|CE|} = \left(\frac{|AB|}{|AC|}\right)^2$$

The proof of this is the same that we used.

Problem 11

Find all positive integer solutions to $m^{n+1} = 2^m + n^2$.

Solution

The only solutions are $(m, n) = (2, 2)$ and $(3, 1)$. It's easy to check these are solutions, and we show these are the only ones by splitting into two cases.

Case 1: If n is *odd*, we get

$$(m^{\frac{n+1}{2}} - n)(m^{\frac{n+1}{2}} + n) = 2^m$$

where both factors on the LHS are integers since n is odd. Hence they must both be powers of 2. But also they differ by $2n$, and n is odd, so their difference is even but not divisible by 4. The only option is then than the smaller one is exactly 2, so we get

$$m^{\frac{n+1}{2}} - n = 2$$

Clearly if $m = 1$ we have no solutions, so assume $m \geq 2$. Then have

$$n + 2 = m^{\frac{n+1}{2}} \geq 2^{\frac{n+1}{2}}$$

If $n = 5$ this is not true, and by induction it doesn't hold for any larger odd n either, so the only options left for odd n are $n = 1, 3$. Only $n = 1$ gives a solution, namely $(m, n) = (3, 1)$.

Case 2: If n is *even*, clearly m is even too so we write $m = 2k$ and $n = 2^\alpha r$ where r is odd (but k can be odd or even). Then get

$$2^{n+1}k^{n+1} = 2^m + 2^{2\alpha}r^2$$

If $2\alpha \neq m$, the RHS is divisible by 2 exactly $min(m, 2\alpha)$ times. On the other hand, the LHS is divisible by 2 at least $n + 1 = 2^\alpha r + 1 \geq 2^\alpha + 1 > 2\alpha$ times, where the final inequality follows by induction on α with base case $\alpha = 1$. This is a contradiction, so we get $2\alpha = m$. Hence $n^2 = 2^{2\alpha}r^2 = 2^m r^2 \geq 2^m$ so we get for $n \geq 5$

$$2^m + n^2 \leq 2n^2 < 2^{n+1} \leq m^{n+1}$$

where the middle inequality follows by induction on n with base case $n = 5$. It remains to check the cases $n = 2, 4$, and only $n = 2$ gives a solution, namely $(m, n) = (2, 2)$.

Problem 12

Is it true that for every finite group G there exists a subset of $\mathbb{R}^n$ (for some n) whose symmetry group is isomorphic to G? *(The symmetry group of a subset of $\mathbb{R}^n$ is the group of isometries sending the subset to itself).*

Solution

By Cayley's theorem, every finite group is isomorphic to a subgroup of S_n for some n. Hence it's enough to consider subgroups of S_n.

Consider the group H of isometries of $\mathbb{R}^n$ which permute the coordinates. There is one such isometry ϕ_σ for each $\sigma \in S_n$, and it's defined by

$$\phi_\sigma(x_1, ..., x_n) = (x_{\sigma^{-1}(1)}, ..., x_{\sigma^{-1}(n)})$$

These together form a subgroup of isometries of $\mathbb{R}^n$ which is isomorphic to S_n (since $\phi_\sigma\phi_\tau = \phi_{\sigma\tau}$ so $\sigma \mapsto \phi_\sigma$ is a homomorphism from $S_n \to H$, and it has trivial kernel). Hence it's enough to show that for every subgroup $G \leq H$, there is some subset of $\mathbb{R}^n$ whose symmetries are given exactly by the elements of G (restricted to that subset - a detail we will skip over in the rest of this solution).

Let X be the subset of $\mathbb{R}^n$ consisting of the points which have all coordinates equal to 0, except one coordinate which is 1 (ie the points $(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, 0, ..., 1)$). Clearly all the isometries in H send X to itself, and so are symmetries of X. Conversely, these are all the symmetries of X, since any symmetry of X is a bijection from X to itself, and H contains all bijections from X to itself. So $H \simeq S_n$ is the symmetry group of X.

Next, let $Y = G \cdot (2, 4, ..., 2n)$ be the set of points that $(2, 4, ..., 2n)$ is sent to by some isometry in G.

We claim that the symmetry group of $X \cup Y$ is exactly G. Indeed:

- Any symmetry of this set must send X to itself, since the distance between any two points in X is $\sqrt{2}$ but the distance from a point in Y to any other point in $X \cup Y$ is at least 2. Hence any symmetry of $X \cup Y$ must be a symmetry of X, and so the symmetry group of $X \cup Y$ is a subgroup of H.
- Furthermore, every isometry in G sends Y to itself, since any point $y \in Y$ is equal to $\phi_\sigma(2, 4, ..., 2n)$ for some $\phi_\sigma \in G$, and hence for every $\phi_\tau \in G$ we have

$$\phi_\tau(y) = \phi_{\tau\sigma}(2, 4, ..., 2n) \in Y$$

- No element of $H \setminus G$ sends Y to itself, since every $\phi_\tau \in H$ sends $(2, 4, ..., 2n) \in Y$ to a different point, and so since the points in Y are all images of $(2, 4, ..., 2n)$ under some $\phi_\sigma \in G$, they can't also be images under some $\phi_\tau \in H \setminus G$.
- Finally, no element of G fixes every point of $X \cup Y$ (indeed, no element of H fixes all elements of X), so all elements of G are genuinely different symmetries even when restricted to $X \cup Y$.

We have now established that every symmetry of $X \cup Y$ is in H, that none is in $H \setminus G$, that all the elements of G are symmetries of $X \cup Y$, and that none of them become the identity when restricted to $X \cup Y$. Hence the group of symmetries of $X \cup Y$ is exactly G, so we are done.

Remark 1: Note that we can formulate the above proof by thinking of S_n as acting on $\mathbb{R}^n$ by permuting coordinates, instead of immediately switching to the point of view where we talk about isometries from the start. The argument is exactly the same, but it's nice noting this analogy. The set Y is then exactly the orbit of $(2, 4, ..., 2n)$ when acting by G.

Remark 2: The points in X are actually all in an affine subspace of dimension $n - 1$ (they form a regular $(n - 1)-$simplex), so we could have done the same proof in $\mathbb{R}^{n-1}$ instead. The points would have to become a bit less explicit though (or have much messier expressions), and it would make it bit harder to rigorously prove that there is a point y which is not fixed by any of the symmetries of the simplex (corresponding to $(2, 4, ..., 2n)$ above), although it's sort of obvious that it exists.

Problem 13

In the land far, far away two teams are competing - the red team and the blue team. There are n cities, some pairs of which are connected by roads. In the beginning, the roads don't belong to anyone. The teams then take turns picking a road that has not yet been picked, and colour it with their own colour. The red team makes the first pick.
If at any point it's possible to travel between any pair of cities only using blue roads, the blue team wins. If all the roads have been picked (by some team) without the blue team achieving this, the red team wins.
Prove that the blue team can guarantee a win if and only if it's possible to split the roads into two groups, such that within each group it's possible to travel between any pair of cities.

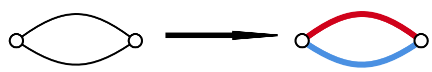
Solution

We will think of the cities as nodes in a graph, with the roads being edges connecting them. If it's possible to get between any pair of nodes only using the edges in a certain group, we say that group of edges is *spanning*. A group of edges is (by definition) spanning if the graph with only these edges is connected. We have two things to show.

The edges can be split into two spanning groups $\implies$ Blue can guarantee a win

Assume the edges can be split into two spanning groups. We show by induction on the number of nodes that the blue team can guarantee a win. In our induction we will allow multiple edges between the same pair of nodes (this is a more general case, so that's fine).

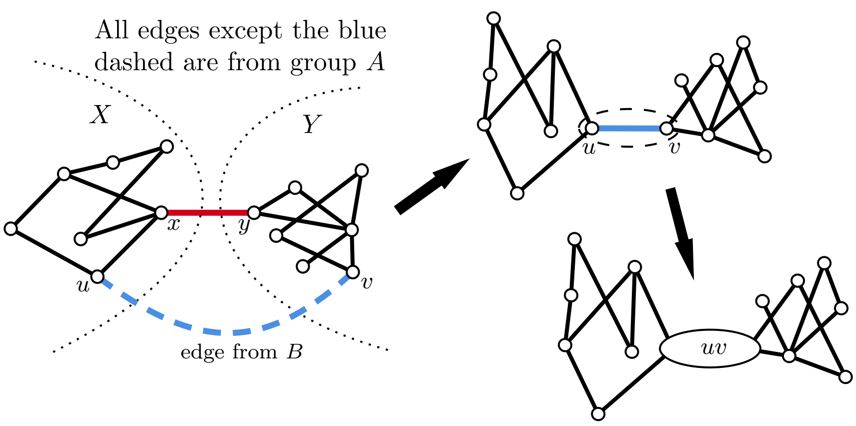
Base case: There are $n = 2$ nodes. The only way in which the edges could be split into two spanning groups is if there are at least two different edges between the nodes.



In this case the blue team can win: red starts by picking one of the edges, after which blue picks the other one (and hence ensures that it's possible to travel between the nodes along a blue edge).

Induction step: By assumption the edges can be split into two spanning groups A and B. The red team makes the first move. Say they colour an edge xy in group A. The graph that only contains the edges from group A is connected, so removing the edge xy splits it into at most two different components - say X is the component containing x and Y is the component containing y. But the graph containing only the edges from group B is also connected, and must hence contain an edge connecting a node $u \in X$ to a node $v \in Y$ (note that $X \cup Y$ is all of the nodes in the graph).

In the left figure below what has been described so far has been drawn. The solid lines are all the edges from group A. If the red edge xy is removed, the graph is split into two components X and Y. Furthermore, there is an edge uv in group B that goes between X and Y (blue dashed line in the figure).



Now let the blue team pick the edge uv (from B), as seen in the upper right figure above (in which all edges from A are drawn *expect* xy, together with one edge uv from B). We can consider u and v

as one node, remove the edge between them and let the remaining edges that connects one of them to a third node w instead connect the node uv to the node w (this has been drawn in bottom right figure above). We then get a graph with $n-1$ nodes. The edges in group A *without* xy are spanning for this new graph, since u was in X and v was in Y. The edges in B are clearly also spanning for the new graph. In other words, the blue team can win in this graph, by induction. But this means that blue can play in the original graph such that it's possible to go from u *or* v to any node, using only blue edges. Recall that blue picked the edge uv in their first move, so then it's possible to get between any pair of nodes using only blue edges. Hence blue can guarantee a win in the original graph as well, so we've completed the induction.

Blue can guarantee a win $\implies$ The edges can be split into two spanning groups

We will show that it's possible to split the edges into two spanning groups. Let the red team "steal" the blue teams strategy, in the following way:

1. In the first move they pick any edge.
2. In every subsequent move, they pretend that the first edge they picked has not been picked yet, and play as the blue team would have played in the corresponding situation (note that since they pretend that the first move was never made, it will look to them as if they are the second team to move).

The only thing which could prevent them from playing exactly as the blue team would have played in the corresponding situation is if they want to pick the first edge again, despite it already having been picked (they are not allowed to pick it twice). In this case, they can simply pick any other edge, and in the future pretend that this is the edge that hasn't been picked yet.

Since the blue team can win (ie guarantee that the edges in their own colour forms a spanning group when all the edges have been picked), the red team will by copying the blue team's strategy as described guarantee that the red edges form a spanning group when all the edges have been picked. But the blue team can guarantee that the blue edges form a spanning group *in the same round of the game* - since this is possible by assumption regardless of how red plays.

We have shown that it's possible to split the edges in two groups (a red and a blue group) such that both groups are spanning. Hence we are done.

Problem 14

An integer n is given and the numbers $1, 2, 3, ..., n$ are written on the board. Kevin wants to pick k of them, and erase the rest, in such a way that no sum of some numbers left on the board is a perfect power. What is the largest k for which he can do this? *The organisers don't know the answer, and will give points for both upper and lower bounds. The better your bounds are asymptotically, the more points you get!*

A lower bound

For a given prime p, if we pick the numbers $p, 2p, 3p, ..., kp$ such that $\frac{k(k+1)}{2} < p \iff k < \frac{-1+\sqrt{8p+1}}{2}$, every sum of some subset of the numbers is divisible by p, but no sum is divisible by p^2. Hence it can't be a perfect power.

The largest of the numbers must be smaller than n, so for this to work we want to pick p such that

$$p \cdot \left\lfloor \frac{-1+\sqrt{8p+1}}{2} \right\rfloor \leq n$$

It's enough that

$$p\sqrt{8p+1} < 2n \quad \Longleftarrow \quad (p+1)^3 < \frac{n^2}{2} \quad \Longleftarrow \quad p < \left(\frac{n}{\sqrt{2}}\right)^{2/3} - 1$$

By Bertrand's postulate, there is a prime between m and $2m$ for every m, so we can find a prime between $\frac{1}{2}\left(\frac{n}{\sqrt{2}}\right)^{2/3} - \frac{1}{2}$ and $\left(\frac{n}{\sqrt{2}}\right)^{2/3} - 1$, and hence with the above strategy we can always find at least

$$\left\lfloor -\frac{1}{2} + \sqrt{\left(\frac{n}{\sqrt{2}}\right)^{2/3} - \frac{3}{4}} \right\rfloor$$

numbers. Asymptotically, this is $cn^{1/3}$ for some constant c (here we showed that $c = 2^{-1/6} \approx 0.9$ works).