HÖJDPUNKTEN
Ung Vetenskapssport
Time: 3h
Tools: Only pen, eraser, compass and ruler

Gymnasietävling, 11th of March 2023

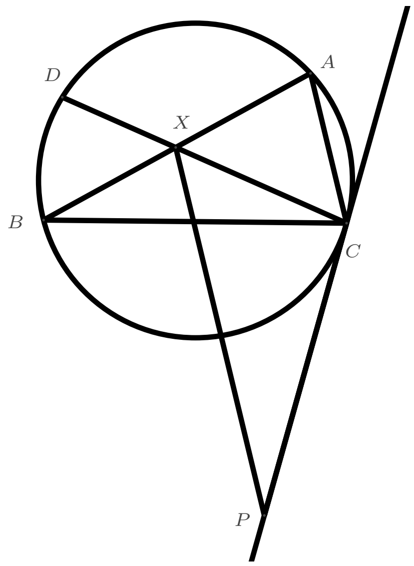
Problem 1. There are 100 people sitting in a circle. We know that some of them are liars who always lie, and some of them are truth tellers who always tell the truth, but we do not know who is what. Everyone in the circle says that they are sitting next to at least one liar. What is the smallest and largest possible number of liars in the circle?

Problem 2. Determine all integers n for which $n^4 - 3n^2 + 9$ is a prime number.

Problem 3. Determine all pairs of positive integers (m, n) such that it's possible to build a rectangle using m vertical dominoes and n horizontal dominoes. A domino is made up of two squares of the same size, glued together along one of their edges. You may not rotate the tiles and they are all the same size.

Problem 4. The students in the UVS-village live in n skyscrapers which are evenly spaced along the main street. One student lives in the first house, two students live in the second house, and so on, until house n where n students live. When it's time to organise a big maths competition, the organisers want to know where to host it in order to minimise the total travel distance for all the students. Which house should they choose? Note that the answer may depend on n.

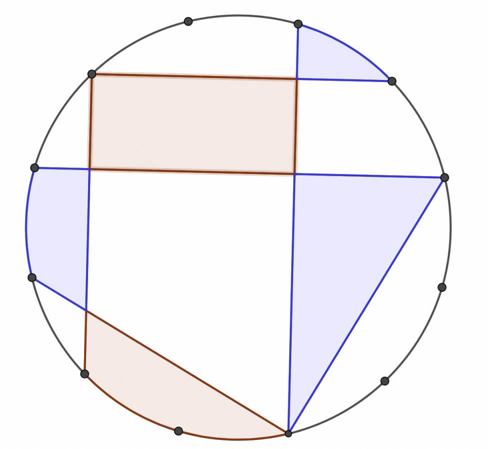
Problem 5. Let ABC be a triangle, and let (ABC) be its circumcircle. Let P be a point outside the circle such that PC is tangent to (ABC), and $|PC| = |BC|$. Let X be a point on the chord AB such that PX is parallel to AC, and let the line CX intersect (ABC) in the point D. Show that $|BD| = |DX|$.



Problem 6. Let $n \leq 100$ be a positive integer. We calculate the numbers $0 \cdot n$, $1 \cdot n$, ..., $100 \cdot n$ in this order, and write down their remainders when dividing by 101 in a row. For how many pairs of adjacent numbers in the row is the left one bigger than the right one?

Problem 7. The numbers $1, 2, 3, ..., 1000$ are written on the board. Kevin picks 12 of them, and erases the rest. He then notes that no sum of some numbers left on the board is a perfect power. Is this possible?

Problem 8. Prove that the sum of the blue areas is equal to the sum of the red areas. The figure is a circle and the points are evenly spaced.



Problem 9. Determine the smallest positive integer n such that if the numbers $404, 405, ..., n$ are divided into two groups, there are always three distinct numbers x, y, z that are in the same group such that $x + y = z$?

Problem 10. Let S be a set of $2^k + 1 \geq 3$ different positive integers. We call a prime *nice* if it divides the sum of two numbers in S. Show that there are at least $k + 1$ nice primes.

Problem 11. In the land far, far away two teams are competing - the red team and the blue team. There are n cities, some pairs of which are connected by roads. In the beginning, the roads don't belong to anyone. The teams then take turns picking a road that has not yet been picked, and colour it with their own colour. The red team makes the first pick.
If at any point it's possible to travel between any pair of cities only using blue roads, the blue team wins. If all the roads have been picked (by some team) without the blue team achieving this, the red team wins.
Prove that the blue team wins if and only if it's possible to split the roads into two groups, such that within each group it's possible to travel between any pair of cities.